Lecture 11: Gronwall's Inequality

Consider the O.D.E.

$$\dot{x} = f(t,x)$$
$$x(0) = x_0$$

$f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with constant K.
Let $u_0, v_0 \in B_r(x_0)$ and $u(t), v(t)$ be their corresponding solutions. Define

$$\varphi(t) = |u(t)-v(t)|^2$$
$$= \langle u(t)-v(t), u(t)-v(t)\rangle$$

$$\Rightarrow \frac{d\varphi}{dt} = \langle \dot{u}(t)-\dot{v}(t), u(t)-v(t)\rangle + \langle u(t)-v(t), \dot{u}(t)-\dot{v}(t)\rangle$$

$$= 2\langle \dot{u}(t)-\dot{v}(t), u(t)-v(t)\rangle$$

$$\Rightarrow \frac{d\varphi}{dt} \leq \left|\frac{d\varphi}{dt}\right| = 2|\langle \dot{u}(t)-\dot{v}(t), u(t)-v(t)\rangle|$$

$$\Rightarrow \frac{d\varphi}{dt} \leq 2|\dot{u}(t)-\dot{v}(t)|\cdot|u(t)-v(t)|$$

$$= 2|f(t,u(t))-f(t,v(t))|\cdot|u(t)-v(t)|$$
$$\leq 2K|u(t)-v(t)|^2$$
$$= 2K\varphi$$

$$\Rightarrow \frac{d\varphi}{dt} - 2K\varphi \leq 0$$

$$\Rightarrow \frac{d}{dt}(e^{-2Kt}\varphi) \leq 0.$$

*Does it follow that

$$\varphi \leq e^{2Kt}\varphi_0$$
$$\Downarrow$$
$$\underbrace{|u(t)-v(t)| \leq e^{Kt}|u(t_0)-v(t_0)|}$$

Measurement of sensitive dependence on initial Conditions.

*Need to show that

$$\frac{d}{dt}(e^{-2Kt}\varphi) \leq 0 \Rightarrow \varphi \leq e^{2Kt}\varphi_0$$

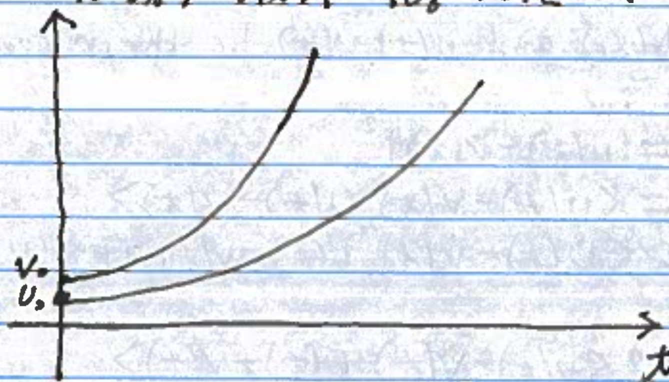
Why Care?

1. $\dot{x} = \lambda x, \quad \lambda > 0$

$\Rightarrow x(t) = x_0 e^{\lambda t}$

If we have two solutions $u(t), v(t)$

$|u(t) - v(t)| = |u_0 - v_0| e^{\lambda t}$.



If $\lambda = 2$, $|u_0 - v_0| = 10^{-16}$. How long before error is 1.

$\Rightarrow 1 = 10^{-16} e^{\lambda t}$

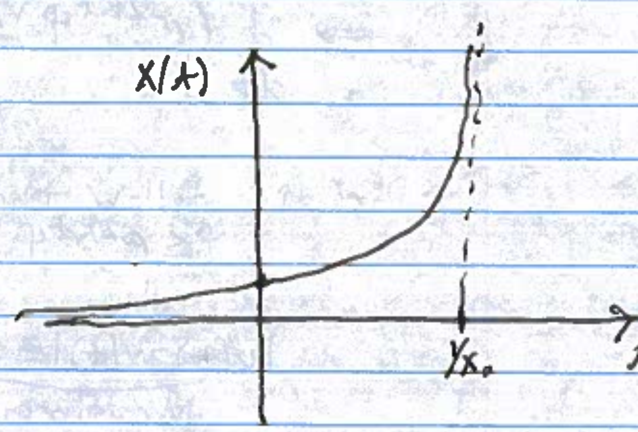
$10^{16} = e^{\lambda t}$

$t = 16 \cdot \ln(10)$

$\approx 32 \rightsquigarrow$ exceptionally fast.

2. $\dot{x} = x^2$

$$\int_{x_0}^{x} \frac{1}{x^2} = \int_0^t dt$$



$$-\frac{1}{x}\Big|_{x_0}^{x} = t$$

$$-\frac{1}{x} + \frac{1}{x_0} = t$$

$$-1 + \frac{x}{x_0} = tx$$

$$\Rightarrow x\left(\frac{1}{x_0} - t\right) = 1$$

$$\Rightarrow x = \frac{x_0}{1 - x_0 t} \longrightarrow \text{blows up at time } t = 1/x_0!$$

3. Lorenz Equations:

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - xz - y$$
$$\dot{z} = xy - bz$$

$(0,0,0)$ is a fixed point

Near $(0,0,0)$:

$$\dot{\vec{x}} = f(\vec{x})$$
$$\approx f(0) + \nabla f \vec{x}$$
$$\approx \left.\begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}\right|_0 \vec{x}$$
$$= \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

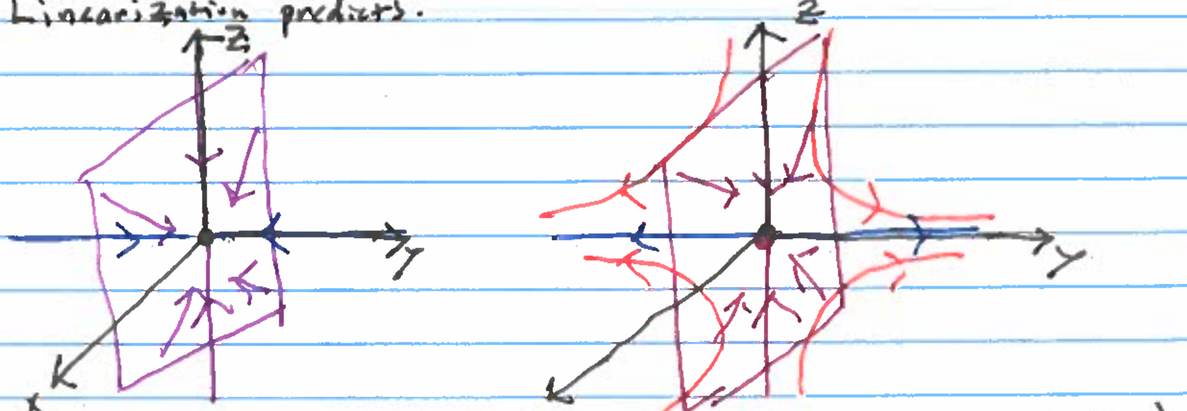
One eigenvalue is $-b$ with eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The other eigenvalues are

$$\lambda = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4(\sigma - \sigma r)}}{2}$$
$$= \frac{-(\sigma+1) \pm \sqrt{\sigma^2 - 2\sigma + 1 + 4\sigma r}}{2}$$
$$= \frac{-(\sigma+1) \pm \sqrt{(\sigma-1)^2 + 4\sigma r}}{2}$$

Two useful forms:

$$\boxed{\lambda = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}}{2}, \quad \lambda = \frac{-(\sigma+1) \pm \sqrt{(\sigma-1)^2 + 4\sigma r}}{2}}$$

Linearization predicts:



$r<1$ $\qquad$ $r>1$ (exponential sep.)