Lecture 12: Flows

*Read Section 4.1!

Dynamical Systems

- An evolution rule that defines a trajectory as a function of a single parameter (time) on a set of states (phase space)

$$\varphi_t : M \to M$$

rule time state space.

$\Gamma^+_x = \{\varphi_t(x) ; t \geq 0\}$ ← forward orbit

$\Gamma^-_x = \{\varphi_t(x) ; t \leq 0\}$ ← pre orbit

$\Gamma_x = \Gamma^+_x \cup \Gamma^-_x$ ← orbit

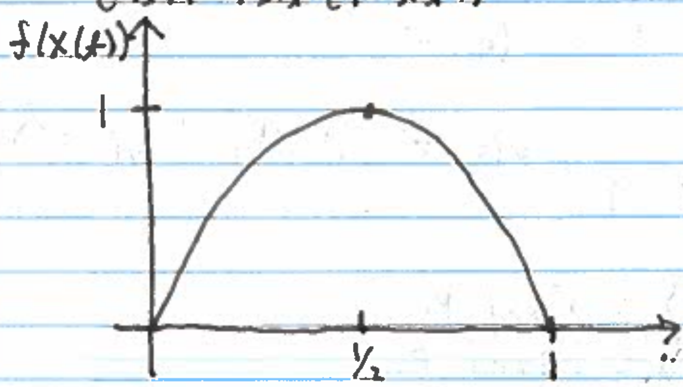
Example:

$M = [0,1]$

$t \in \mathbb{Z}$

$x(t+1) = x(t)(1-x(t)) = f(x(t))$ | $x(t) = \frac{1}{2}\left(1 \pm \sqrt{1 - x(t+1)}\right)$

$(x_{t+1} = 4x_t(1-x_t))$



$$\Downarrow$$

$$\varphi_t(x) = f^t(x)$$

$\Gamma^+_{1/4} = \{1/4, 3/4, 3/4, \ldots\} = \{1/4, 3/4\}$

$\Gamma^-_{1/4} = \{\ldots, \frac{1}{2}(1 - \sqrt{3}/2), 1/4\}$

$\Gamma_x = \{\ldots, \frac{1}{2}(1-\sqrt{3}/2), 1/4, 3/4, 3/4, \ldots\} = \{\ldots, \frac{1}{2}(1-\sqrt{3}/2), 1/4, 3/4\}$.

* An orbit is an <u>equilibrium</u> if
$$\Gamma_x = \{x\}.$$

* A point x is on a <u>periodic orbit</u> if there exists $T>0$ such that
$$\varphi_T(x) = x$$

* A set Δ is <u>invariant</u> under φ_t if $\varphi_t(\Delta) = \Delta$ for all t. That is, for each $x \in \Delta$, $\varphi_t(x) \in \Delta$.

* A set Δ is <u>forward invariant</u> if $\varphi_t(\Delta) \subset \Delta$ for all $t > 0$.

Example:

$\dot{r} = r(1-r)$. $\quad M = \mathbb{R}^+ \times S^1$

$\dot{\theta} = \sin\theta$ $\quad$ positive or equal to zero reals $\quad$ circle

$r(0) = r_0$

$\theta(0) = \theta_0$

$\varphi_t(r_0, \theta_0) \rightsquigarrow$ solution of O.D.E. at time t given initial condition (r_0, θ_0).

$$\boxed{r(t) = \frac{r_0 e^t}{1 + r_0(e^t - 1)}, \quad \theta(t) = 2\cot^{-1}\left(e^{-t}\cot(\theta_0/2)\right)}$$

$\Gamma_{(1,0)} = \{(1,0)\} \leftarrow$ Fixed Point

$\Gamma_{(1,\pi)} = \{(1,\pi)\} \leftarrow$ Fixed Point

$\Gamma_{(1,\theta_0)} = \{(r,\theta) \in M : r = 1, \ \theta \in (0,\pi)\}$.

$\Gamma_{(0,\theta)} = \{(r,\theta) \in M : r = 0\}$.

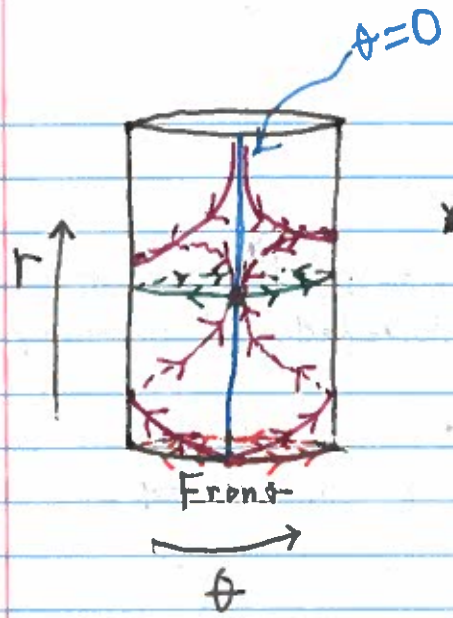
$\Delta = \{(r,\theta) \in M : r > 1\}$ is invariant and forward invariant.

$\Delta = \{(r,\theta) \in M : r < 1\}$ is " "

$\Delta = \{(r,\theta) \in M : r = 1\}$ is " "

$\Delta = \{(r,\theta) \in M : \theta = 0\}$ is " "

$\Delta = \{(r,\theta) \in M : \theta = \pi\}$ is " "



*The solution curves correspond to the orbits.

$\varphi_t : M \to M$ is a parameterized sequence of maps!!

Flows

A complete flow $\varphi_t(x)$ is a one-parameter, differentiable mapping $\varphi : \mathbb{R} \times M \to M$ such that

$$\left.\begin{aligned} &(a)\ \varphi_0(x) \equiv x \\ &(b)\ \varphi_t \circ \varphi_s = \varphi_{t+s} \\ &(c)\ \varphi_t \circ \varphi_{-t} = \varphi_0 = id. \\ &\quad \Rightarrow (\varphi_t)^{-1} = \varphi_{-t} \end{aligned}\right\} \text{group properties.}$$

Vector field associated with flow

$$f(x) = \frac{d}{dt}\varphi_t(x).$$

$\Rightarrow \varphi_t(x)$ is defined by

$$\frac{d}{dt}\varphi_t(x_0) = f(\varphi_t(x_0))$$

~~proof~~:

Let $x(t) = \varphi_t(x_0)$.

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\varphi_{t+\Delta t}(x_0) - \varphi_t(x_0)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\varphi_{\Delta t}(x) - \varphi_0(x)}{\Delta t}$$

$$= \frac{d}{dt}\varphi_t(x) = f(x).$$

Complete Flows

$$\frac{dx}{dt} = f(x)$$

Could have unbounded solutions. How do we rescale so that solutions remain bounded and have equivalent dynamics.

If $x(t)$ is a solution on some maximal interval (α, β). Define

$$y(\tau(t)) = x(t)$$
$$\tau = \int_0^t (1 + |f(x(s))|)\,ds.$$

$$\Rightarrow \frac{d\tau}{dt} = 1 + |f(x(t))| > 0 \Rightarrow \text{1-1 mapping.}$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{f(x)}{1+|f(x(t))|}$$

$$\Rightarrow \boxed{\frac{dy}{d\tau} = \frac{f(y)}{1+|f(y)|} = F(y)}$$ ~ same dynamics but F is Lipschitz!

F ~ is globally Lipschitz.

Example:

$$\dot{x} = x^2$$
$$x(0) = x_0$$
$$\downarrow$$
$$x(t) = \frac{x_0}{1 - x_0 t}$$

$$\frac{dy}{d\tau} = \frac{y^2}{1+y^2}, \quad \left|\frac{dy}{d\tau}\right| < 1$$
$$\downarrow$$
$y(\tau)$ ~ bounded solution

$$\tau = \int_0^t \left(1 + \left(\frac{x_0}{1 - x_0 t}\right)^2\right) dt = t + \frac{t x_0^2}{1 - t x_0}$$

$\tau \in (-\infty, \infty)$.