Lecture 13: Linearization and Stability

- $\dot{x} = f(x)$
  x^* satisfies $f(x^*) = 0$ (Equilibrium, Fixed Point).
- $\dot{x} = f(x^*) + \nabla f|_{x^*}(x - x^*) + o(\|x - x^*\|)$

Assuming f is analytic.

Recall:
We say $g(x) = o(\|x - x^*\|)$ if
$$\lim_{x \to x^*} \frac{\|g(x)\|}{\|x - x^*\|} = 0$$
(i.e. g goes to 0 faster than $\|x - x^*\|$)

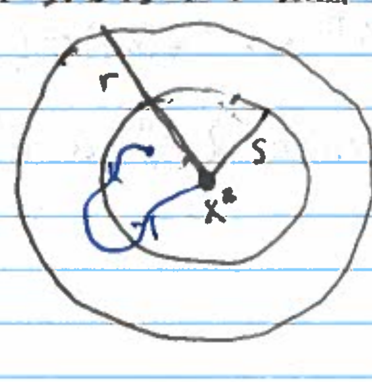
Letting $y = x - x^*$ the linearized O.D.E is given by
$$\dot{y} = \nabla f|_{x^*} y.$$

- x^* is hyperbolic if none of eigenvalues of $\nabla f|_{x^*}$ have 0 real part.

- Sink (source) if eigenvalues have negative (positive) real parts.

- Saddle if hyperbolic but not source or sink.

Stability
- An equilibrium x^* is Lyapunov stable if for every $B_r(x^*)$ there exists $B_s(x^*)$ with $s \le r$ such that if $x \in B_s(x^*)$ then $\varphi_t(x) \in B_r(x^*)$ for all $t \ge 0$.

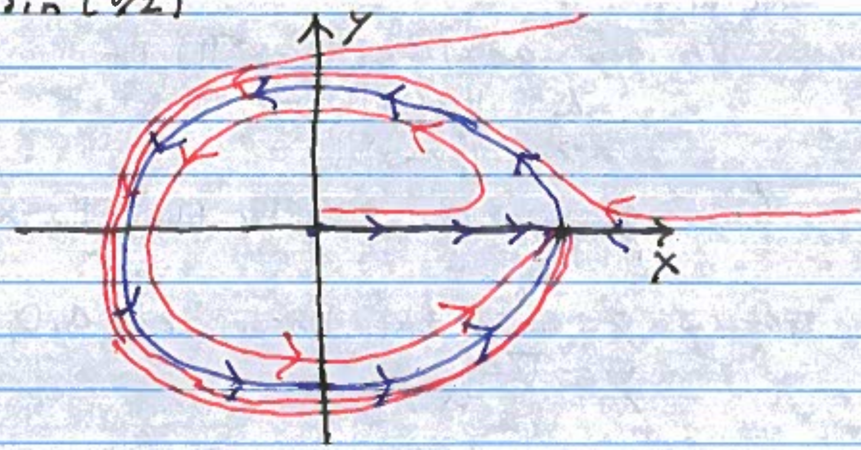


- A fixed point is <u>asymptotically stable</u> if it is Lyapunov stable and there exists $B_r(x^*)$ such that if $x \in B_r(x^*)$ then $\lim_{t\to\infty} x(t) = x^*$.

Example:

$\dot{r} = r(1-r)$

$\dot{\theta} = \sin^2(\theta/2)$



Gronwall's Inequality:

If

$$g(t) \le c + \int_0^t k(s) g(s)\,ds,$$

where $k \ge 0$ then

$$g(t) \le c\, e^{\int_0^t k(s)\,ds}.$$

~~proof~~:

Let $G(t) = c + \int_0^t k(s) g(s)\,ds$. Then,

$$G'(t) = k(t) g(t)$$

$$\le k(t) G(t)$$

$$\Rightarrow G'(t) - k(t) G(t) \le 0$$

$$\Rightarrow e^{-\int_0^t k(s)\,ds}(G'(t) - k(t)G(t)) \le 0$$

$$\Rightarrow \frac{d}{dt}\left(\exp\left(-\int_0^t k(s)\,ds\right) G(t)\right) \le 0$$

$$\Rightarrow G(t) \le G(0) \exp\left(-\int_0^t k(s)\,ds\right) = c \exp\left(-\int_0^t k(s)\,ds\right).$$

∎

Main Theorem

Let $f \in C^1$ and have an equilibrium x^* such that the linear system

$$\dot{y} = \nabla f(x^*) y$$

is asymptotically stable. Then x^* is asymptotically stable for the system

$$\dot{x} = f(x).$$

~~proof~~:

By construction, $\dot{x} = f(x)$ is equivalent to

$$\dot{y} = \nabla f(x^*) y + g(y) = Ay + g(y),$$

where $g(y) = o(y)$.

$$\Rightarrow y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} g(y(s))\, ds$$

For simplicity if A is diagonalizable

$$\|e^{tA}\| = \|V e^{t\Lambda} V^{-1}\|$$
$$\leq \|V\| \|e^{t\Lambda}\| \cdot \|V^{-1}\|$$
$$\leq e^{-t\lambda^*}$$

where $\lambda^* = \min\{|\lambda_1|, \ldots, |\lambda_n|\}$. Consequently,

$$\|y\| \leq \|y_0\| e^{-t\lambda^*} + e^{-t\lambda^*} \int_0^t e^{s\lambda^*} \|g(y(s))\| ds.$$

If $|y_0| < \delta$ then $\|g(y(s))\| < \varepsilon \|y\|$. By continuity $\|y(t)\| < \delta$ for $t < \tau$ for some τ. Therefore,

$$e^{t\lambda^*} \|y\| \leq \|y_0\| + \varepsilon \int_0^t e^{s\lambda^*} \|y\| ds$$

Define $\gamma = e^{t\lambda^*} \|y\|$. By Gronwall's inequality it follows that:

$$\gamma \leq \|y_0\| e^{\varepsilon t}$$
$$\Rightarrow \|y\| \leq \|y_0\| e^{-(\lambda^* - \varepsilon)t}$$
$$\Downarrow$$
$$\lim_{t \to \infty} \|y\| = 0.$$

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