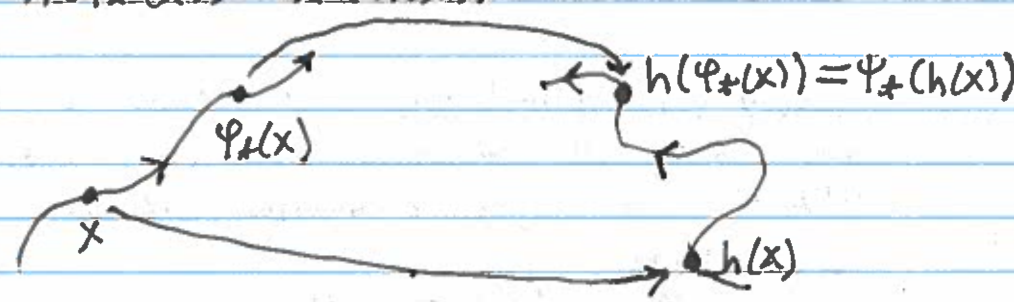
Lecture 14: Topological Conjugacy and Equivalence

Homeomorphisms - A map $h: A \to B$ is a homeomorphism if it is continuous, bijective, and has a continuous inverse.

Diffeomorphisms - A map $h: A \to B$ is a diffeomorphisms if it is a C^1 bijective map with a C^1 inverse.

Topological Conjugacy - Two flows $\varphi_t: A \to A$ and $\psi_t: B \to B$ are conjugate if there exists a homeomorphism $h: A \to B$ such that for each $x \in A$ and $t \in \mathbb{R}$

$$h(\varphi_t(x)) = \psi_t(h(x)).$$



Example:

$$\dot{x} = -x$$
$$\varphi_t(x) = xe^{-t}$$

Let $h(x) = x^3$, $\Rightarrow h(\varphi_t(x)) = x^3 e^{-3t}$.
We use this to define a new flow. Let

$$y = h(x) = x^3$$
$$\psi_t(y) = ye^{-3t}$$
$$\Downarrow$$
$$\frac{d\psi_t}{dt} = -3ye^{-3t} = -3\psi_t$$

Equivalent vector field:

$$\dot{y} = -3y.$$

Consequences:

1. If x^* is an equilibrium of φ_t we have

$$\varphi_t(x^*) = x^*$$
$$\Rightarrow h(\varphi_t(x^*)) = h(x^*)$$
$$\Downarrow$$
$$\psi_t(h(x^*)) = h(x^*)$$

2. Periodic orbits are preserved

$$\varphi_{t+T}(x_0) = \varphi_t(x_0)$$
$$\Downarrow$$
$$\psi_{t+T}(h(x_0)) = h(\varphi_{t+T}(x_0)) = h(\varphi_t(x_0)) = \psi_t(h(x_0))$$

3. Parametrization is preserved...

Topological Equivalence - Two flows $\varphi_t : A \to A$ and $\psi_t : B \to B$ are topologically equivalent if there exists a homeomorphism $h : A \to B$ and surjective, monotone increasing in t map $\tau : A \times \mathbb{R} \to \mathbb{R}$ such that

$$h(\varphi_{\tau(x,t)}(x)) = \psi_t(h(x)).$$

$\Rightarrow$ Maps orbits to orbits

$\Rightarrow$ Preserves direction of time.

Example:

Consider the flows

$$(x \in \mathbb{R}^+) \quad \varphi_t(x) = xe^{-t}, \qquad \psi_t(y) = \frac{y}{1+ty} \quad (y \in \mathbb{R}^+)$$

$$\frac{d\varphi_t}{dt} = -xe^{-t} = -\varphi \qquad \frac{d\psi_t}{dt} = \frac{-y^2}{(1+ty)^2} = -\psi^2$$

$$\dot{x} = -x \qquad \dot{y} = -y^2$$

Let $h(x) = x$, $\tau(x,t) = \ln(1+xt)$

$$h(\varphi_\tau(x)) = xe^{-\ln(1+xt)} = \frac{x}{1+xt} = \psi_t(h(x)).$$

Diffeomorphic Flows

$\varphi_t(x) = xe^{-t}$ flow generated by $\dot{x} = -x$.

Let $h: \mathbb{R} \to (-1,1)$ be defined by

$$h(x) = \tanh(x)$$

If φ_t is topologically conjugate to ψ_t via h then

$$h(\varphi_t(x)) = \psi_t(h(x))$$

$$\Rightarrow h^{-1}(\psi_t(y)) = \varphi_t(h^{-1}(y))$$

$$\Rightarrow \psi_t(y) = h \circ \varphi_t \circ h^{-1}$$

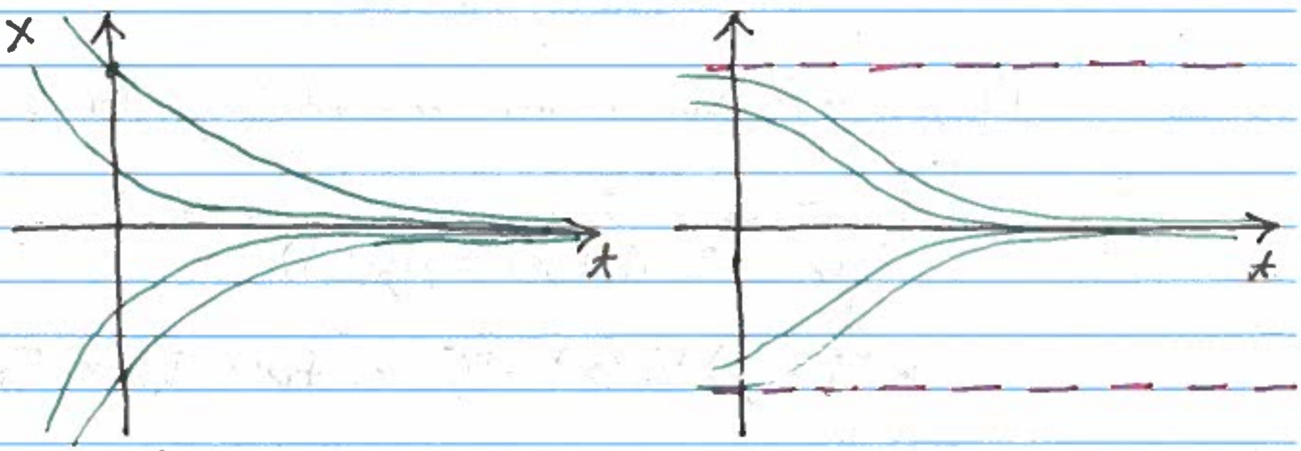
$$\Downarrow$$

$$\psi_t(y) = \tanh(\tanh^{-1}(y)e^{-t})$$

$$\frac{d\psi_t}{dt} = \operatorname{sech}^2(\tanh^{-1}(y)e^{-t})\tanh^{-1}(y)(-e^{-t})$$

$$\rightarrow \left.\frac{d\psi_t}{dt}\right|_{t=0} = -\operatorname{sech}^2(\tanh^{-1}(y))\tanh^{-1}(y)$$

$$\Rightarrow \dot{y} = (y^2-1)\tanh^{-1}(y)$$ (Corresponding O.D.E).



(Compactified the range).

General Procedure:

$\dot{x} = f(x)$, $\dot{y} = g(y)$

$\varphi_t(x) \sim$ flow , $\psi_t(y) \sim$ flow

If diffeomorphic then

$$\frac{d}{dt}\psi_t(y) = g(\psi_t(y))$$

$$\Rightarrow \frac{d}{dt} h(\varphi_t(x)) = g(\psi_t(y))$$

$$\Rightarrow \nabla h \frac{d}{dt}\varphi_t(x) = g(\psi_t(y))$$

$$\Rightarrow \nabla h(\varphi_t(x)) f(\varphi_t(x)) = g(\psi_t(y))$$

Setting $t=0$ we have:

$$\nabla h(x) f(x) = g(y) = g(h(x))$$

$$\Rightarrow \boxed{g(y) = \nabla h(x) f(x).}$$

Theorem - Eigenvalues are preserved by a diffeomorphism!

~~proof~~:

Let x^* be a fixed point of $\dot{x} = f(x)$,

$$\frac{dy}{dt} = \nabla h(x) \cdot f(x) = g(y(x))$$

$$\Rightarrow \nabla_y g(y) \nabla_x h(x) = \nabla_x^2 h(x) \cdot f(x) + \nabla_x h(x) \nabla_x f(x).$$

Evaluating at x^* we have:

$$\nabla_y g(y^*) \nabla_x h(x^*) = \nabla_x h(x^*) \nabla_x f(x^*)$$

$$\Rightarrow \nabla_x f(x^*) = [\nabla_x h(x^*)]^{-1} \nabla_y g(y^*) \nabla_x h(x^*)$$

Similar matrices imply same eigenvalues.

Theorem - The flows φ_t and ψ_t of linear systems $\dot{x}=Ax$, $\dot{y}=By$ are topologically conjugate if A is similar to B.

~~pf~~:

If A and B are similar matrices then there exists a nonsingular matrix H such that

$$A = H^{-1}BH$$

$$\Rightarrow HA = BH$$

The map $h: \mathbb{R}^n \to \mathbb{R}^n$ defined by $h(x)=Hx$ is clearly a homeomorphism.

$$\begin{aligned} h(\varphi_t(x)) &= h(e^{tA}x) \\ &= He^{tA}x \\ &= \left(H + tHA + \frac{t^2}{2}HA^2 + \dots\right)x \\ &= \left(H + tH\cdot H^{-1}BH + \frac{t^2}{2}HH^{-1}B^2H + \dots\right)x \\ &= \left(I\cdot H + tB\cdot H + \frac{t^2}{2}B^2H + \dots\right)x \\ &= e^{tB}Hx \\ &= \psi_t(h(x)). \end{aligned}$$

$\blacksquare$

Theorem - Let x^* be a hyperbolic fixed point of a C^1 vector field $f(x)$ with flow $\varphi_t(x)$. Then there is a neighborhood N of x^* such that φ is topologically conjugate to its linearization.