Lecture 15: Attractors and Basins

Limit Points: A point y is a limit point of $\varphi_t(x)$ if there exists a subsequence t_k such that

$$\varphi_{t_k}(x) \to y.$$

ω-limit set: The collection of all limit points of Γ_x^+ is the ω-limit set of x denoted $\omega(x)$.

* $\alpha(x)$ is the limit in backwards time $t_k \to -\infty$.

Limit cycle: A periodic orbit γ that is the ω-limit set of a point $x \notin \gamma$ is a limit cycle.

* $\omega(x) = \bigcap_{T \geq 0} \overline{\Gamma}_{\varphi_T(x)} \Rightarrow \omega(x)$ is a closed set.

Example:

$$\dot{x} = y \qquad = f(x,y)$$
$$\dot{y} = x - x^3 - \nu y\left(y^2 - x^2 + \tfrac{1}{2}x^4\right) \qquad = g(x,y;\nu)$$

If $\nu = 0$ we have

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

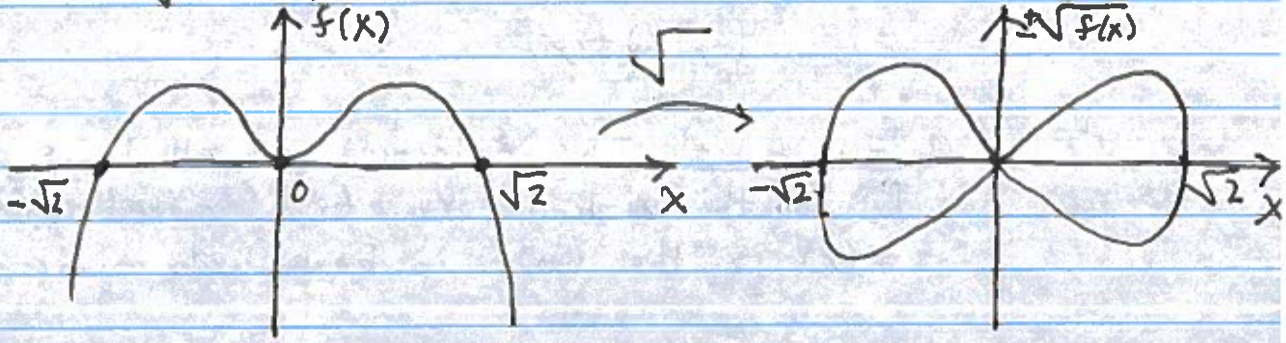
$\Rightarrow \dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial y} \Rightarrow$ Hamiltonian system

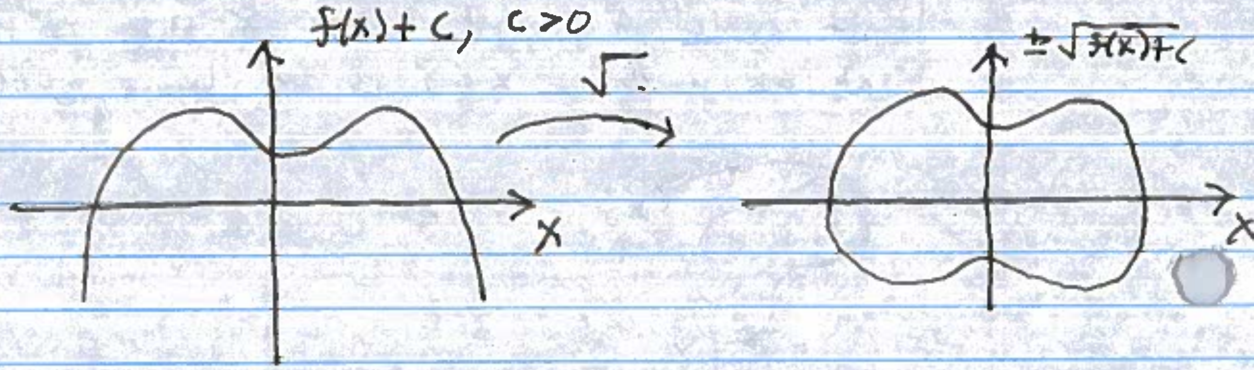
$$H = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

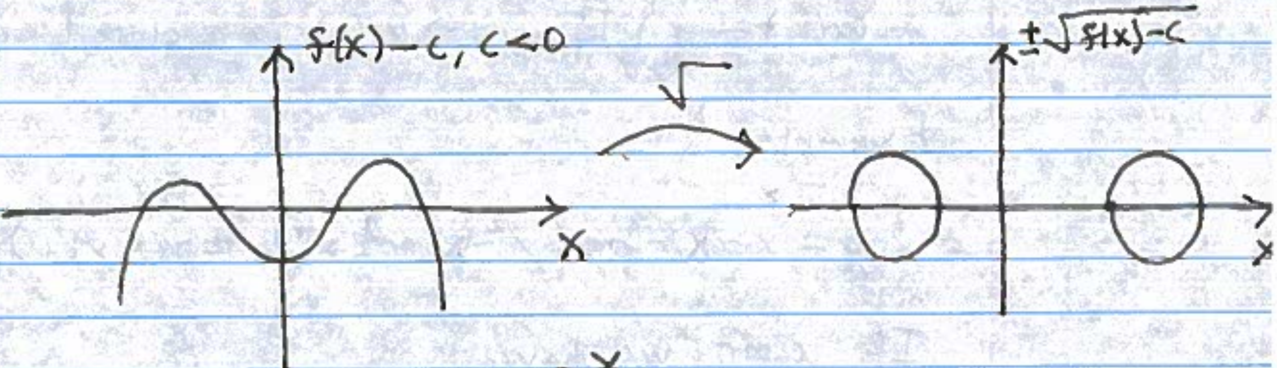
Phase Portrait $\mu = 0$

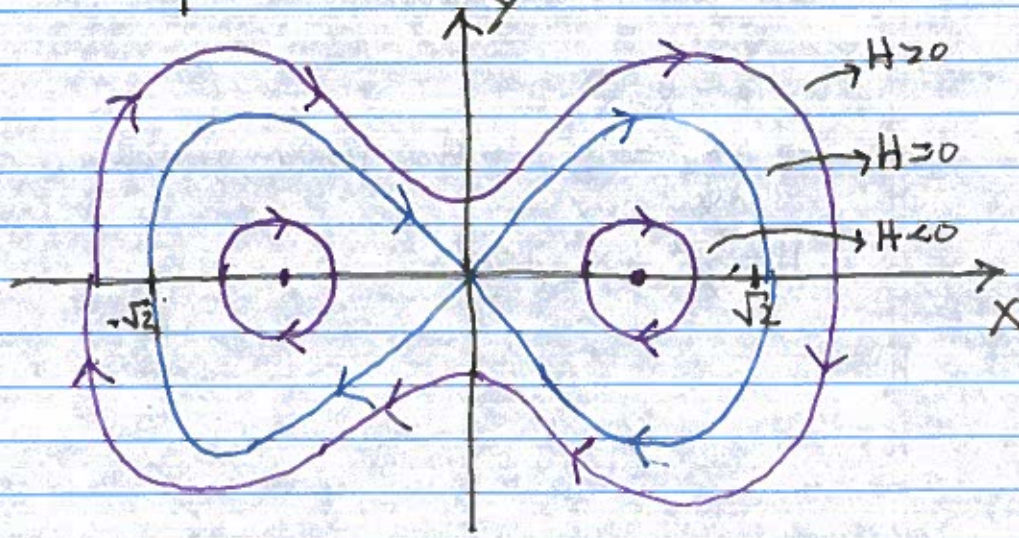
$H = c$

$$\Rightarrow y = \pm\sqrt{\frac{x^2}{2} - \frac{x^4}{4} + c} = \pm\sqrt{f(x) + c}$$







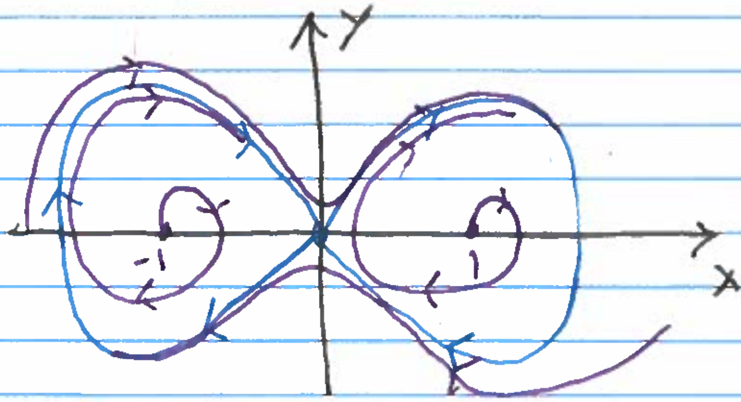


Phase Portrait $\nu > 0$

$$\dot{x} = y$$
$$\dot{y} = x - x^3 - 2\nu y H$$

→ perturbed Hamiltonian.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = -2\nu H$$



What are the ω-limit sets.

- right lobe, Λ_r
- left lobe, Λ_l
- 3 fixed points, x_{-1}^*, x_0^*, x_1^*
- entire figure eight. Λ_8

Stability - An invariant set Λ is stable if for any nbd of N of Λ there is a nbd S of Λ such that all points that start in S stay in N.

Asymptotic Stability - An invariant set Λ is asymptotically stable if it is stable and has a nbd N such that for each $x \in N$, $\rho(\varphi_t(x), \Lambda) \to 0$ as $t \to \infty$.

Trapping region - A set N is a trapping region if it is compact and $\varphi_t(N) \subset \text{int}(N)$ for $t > 0$.

Attracting Set - A set Λ is an attracting set if there is a compact trapping region N such that $\Lambda \subset N$ and $\bigcap_{t \geq 0} \varphi_t(N) = \Lambda$.

Basin of Attraction - $W^s(\Delta)$ = The basin of an attraction for an invariant set Δ is the set of all points x for which $\rho(\varphi_t(x), \Delta) \to 0$ as $t \to \infty$.

- The only attracting set is Δ_8.