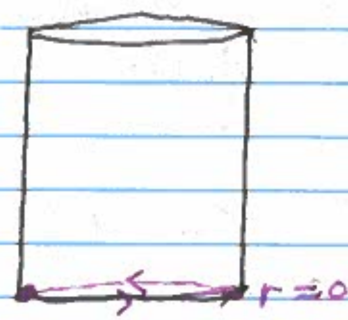
Lecture 16: Periodic Orbits and Poincare Maps

Example:

$$\dot{r} = r(1 + a\cos\theta - r^2) \qquad M = [0,\infty) \times S^1,$$
$$\dot{\theta} = 1$$
$$|a| < 1.$$

1. One periodic Orbit is $r_1 = 0$. The period is given by

$$T = 2\pi$$



$$\int_0^T \frac{d\theta}{dt}\,dt = \int_0^{2\pi} d\theta = 2\pi$$

2. Is r_1 attracting? Taylor expand about $r_1 = 0$, $\theta_1 = t$

Define

$$r = r_1 + \tilde{r} = \tilde{r}$$
$$\theta = \theta_1 + \tilde{\theta} = t + \tilde{\theta}$$

$$\Rightarrow \frac{d\tilde{r}}{dt} = \dot{r}, \quad 1 + \frac{d\tilde{\theta}}{dt} = \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\tilde{r}}{dt} = \tilde{r}(1 + a\cos(t + \tilde{\theta}) - \tilde{r}^2)$$

$$\frac{d\tilde{\theta}}{dt} = 0,$$

$$\Rightarrow \frac{d\tilde{r}}{dt} \approx \tilde{r}(1 + a\cos(t + \tilde{\theta}_0))$$
$$\tilde{\theta} = \tilde{\theta}_0$$

$$\Rightarrow \frac{1}{\tilde{r}}\frac{d\tilde{r}}{dt} = 1 + a\cos(t + \tilde{\theta}_0)$$

$$\Rightarrow \ln\left(\frac{\tilde{r}}{\tilde{r}_0}\right) = t + a(\sin(t + \tilde{\theta}_0) - \sin(t))$$

$$\Rightarrow r(t) = r_0 \exp\left(t + a(\sin(t+\tilde{\theta}_0) - \sin(\tilde{\theta}_0)\right)$$

$$\Rightarrow \tilde{r}(T) = r_0 \underbrace{\exp(2\pi)}_{\text{Floquet multiplier}} \to \text{Floquet Exponent}$$

$$\tilde{\theta}(T) = \theta_0 \exp(0) \to \text{Neutral}$$

If Floquet multiplier $> 1 \Rightarrow$ unstable
If Floquet multiplier $< 1 \Rightarrow$ stable.

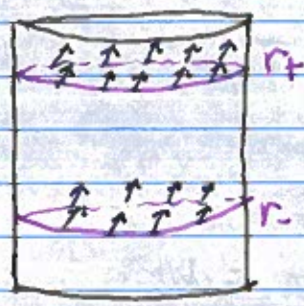
3. Let $r_+ = 2$ and $r_- = \sqrt{1 - a/2}$ then

$$\left.\frac{dr}{dt}\right|_{r_+} < 0 \quad \text{and} \quad \left.\frac{dr}{dt}\right|_{r_-} > 0.$$

Therefore

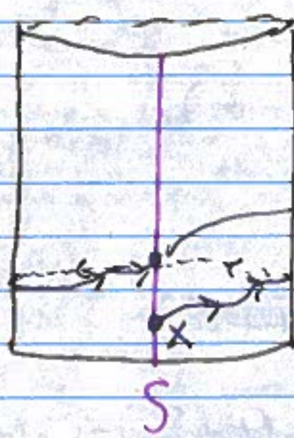
$$N = \{ (r, \theta) : \sqrt{1-a/2} < r < 2 \text{ and } \theta \in S^1 \}$$

is a trapping region with some attracting set Δ.



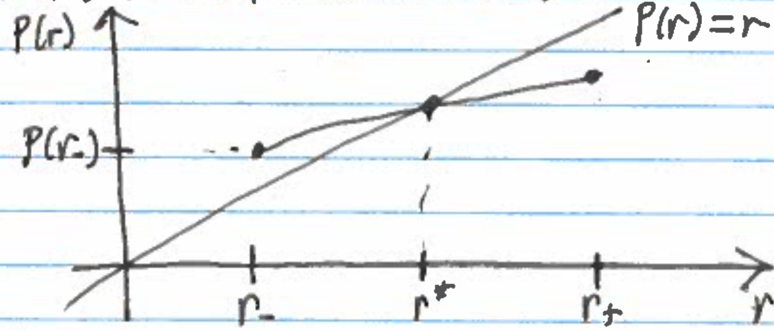
4. Let $S = \theta = 0$, define a map $P : S \to S$ by $\to$ Poincaré return map.

$$P(x) = \varphi_{2\pi}(x)$$



$\to P(x)$ = first time orbit of x crosses S.

5. $P(r_+) < r_+$ and $P(r_-) > r_-$



By the intermediate value theorem there exists r^* such that $P(r^*) = r^*$

$\Rightarrow \omega(r^*) =$ periodic orbit.

6. Is this limit cycle stable?

*let $r^*(t) =$ limit cycle and $\tilde{r}$ a perturbation.

$$\Rightarrow \dot{r}^* + \dot{\tilde{r}} = (r^* + \tilde{r})(1 + a\cos\theta - (r^* + \tilde{r})^2)$$
$$= (r^* + \tilde{r})(1 + a\cos\theta - r^{*2} - 2r^*\tilde{r} - \tilde{r}^2)$$
$$= r^*(1 + a\cos\theta - r^{*2} - 2r^*\tilde{r}) + \tilde{r}(1 + a\cos\theta - r^{*2})$$
$$\Rightarrow \dot{\tilde{r}} = \tilde{r}(-3r^{*2} + 1)$$
$$\Rightarrow \int_{r_0}^{r_f} \frac{1}{\tilde{r}}\, d\tilde{r} = \int_0^{2\pi} \left(-3r^{*2} + 1\right) dt$$
$$= \int_0^{2\pi} \left(-3\left(1 + a\cos\theta - \frac{1}{r^*}\frac{dr^*}{dt}\right) + 1\right) dt$$

$$= -6\pi + 2\pi$$
$$= 4\pi$$
$$\Rightarrow r_f = r_0 e^{-4\pi}$$

Locally, $P(r) = e^{-4\pi} r$

$\Rightarrow$ Floquet Multiplier $= e^{-4\pi}$

Floquet Exponent $= -4\pi$

$\Rightarrow$ Stable.