Lecture 17: Conservative Dynamics

$\dot{x} = f(x)$

$I(x)$ is conserved if

$$\frac{dI}{dt} = \frac{d}{dt} I(x) = \nabla I(x) \cdot \frac{dx}{dt} = \nabla I \cdot f(x) = 0$$

We call I an invariant if $\nabla I \cdot f(x) = 0$.
Classically called integral of motion or conserved quantity.

$$\nabla I \cdot f = 0 \rightarrow \text{PDE for integral of motion.}$$

Example:

$X + Y \xrightarrow{k_1} Y$ (Chemical Reaction).

$Y \xrightarrow{k_2} X$

$$\Rightarrow [\dot{X}] = -k_1 [X]\cdot[Y] + k_2 [Y]$$
$$[\dot{Y}] = k_1 [X]\cdot[Y] - k_2 [Y]$$

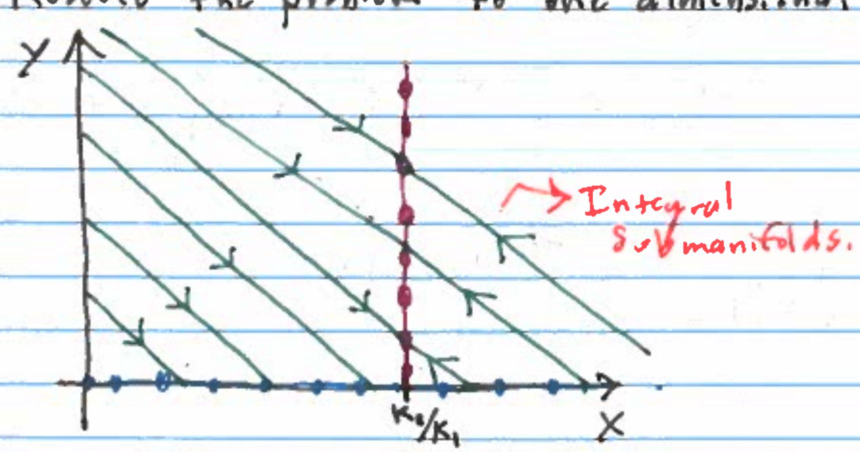
$P = [X] + [Y]$ is conserved.

$$\nabla P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

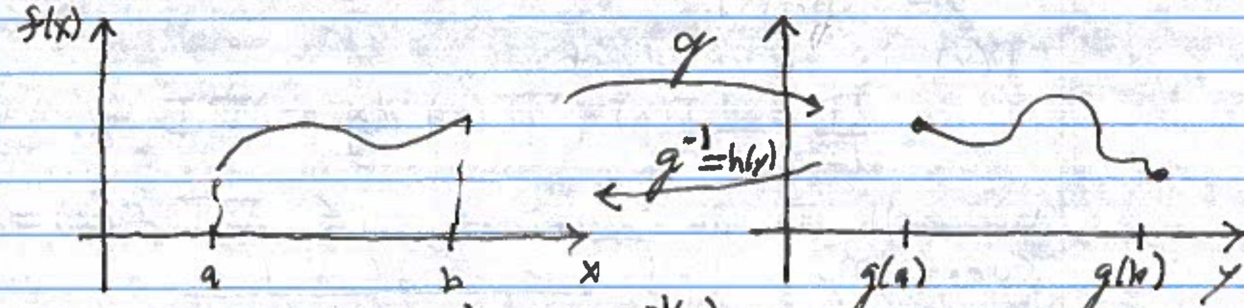
$$\nabla P \cdot f = [\dot{x}] + [\dot{Y}] = 0$$

$$\Rightarrow [\dot{Y}] = k_1 (P - [Y])[Y] + k_2 [Y]$$

* Reduce the problem to one dimensional submanifold



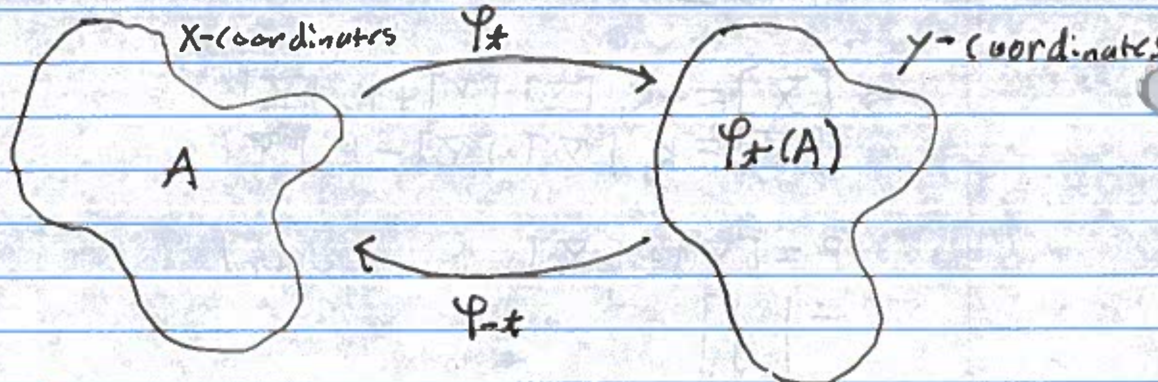
Volume Preserving Flow:



$$y = g(x) \Rightarrow x = g^{-1}(y)$$

$$\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} f(h(y))\frac{dx}{dy}\,dy$$

$$= \int_{g(a)}^{g(b)} f(h(y))\frac{dh}{dy}\,dy$$



$$V(t) = \int_{\varphi_t(A)} dV = \int_A \det(\nabla \varphi_t)\,dV$$

Volume of $\varphi_t(A)$

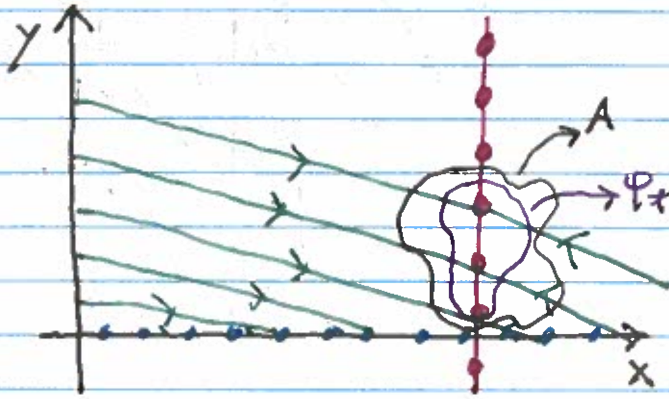
$$= \int_A \left(1 + \nabla\cdot f(x)\,t + o(t)\right)dV$$

$$\Rightarrow \frac{V(t)-V(0)}{t} = \int_A \nabla\cdot f(x)\,dV + \int_A \frac{o(t)}{t}\,dV$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{t=0} = \int_A \nabla\cdot f(x)\,dV$$

Example:

$$\dot{x} = -k_1 xy + k_2 y \qquad \nabla \cdot f = -k_1 y + k_1 x - k_2$$
$$\dot{y} = k_1 xy - k_2 y \qquad \text{(not volume preserving).}$$

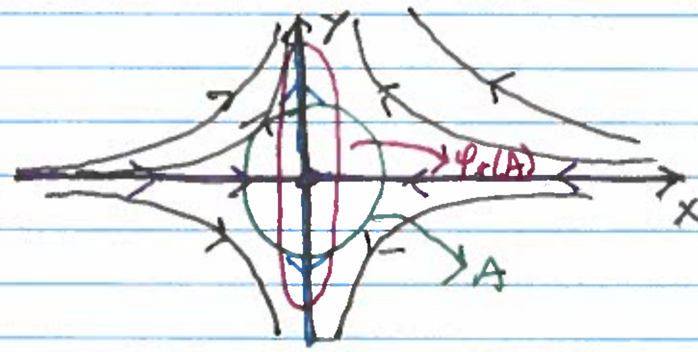


→ Asymptotically stable fixed points cannot preserve volume!

Example:

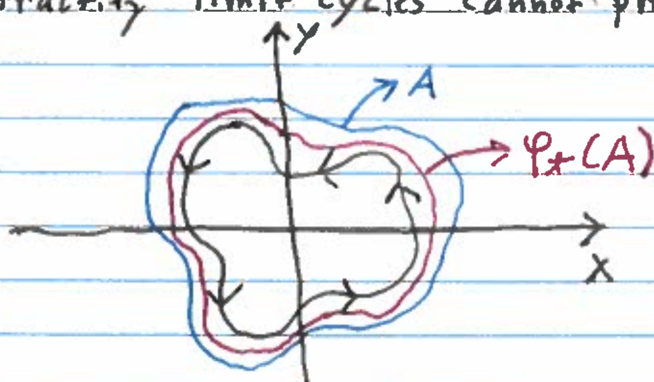
$$\dot{x} = -x$$
$$\dot{y} = y$$

$$\nabla \cdot f = -1 + 1 = 0$$



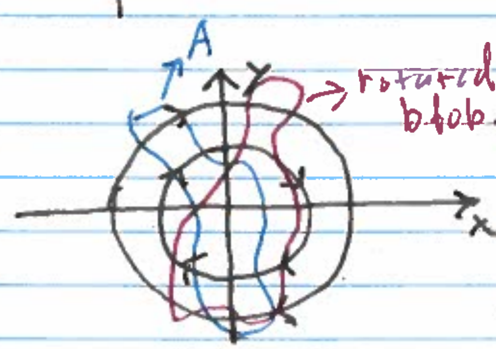
Example:

Attracting limit cycles cannot preserve volume.



Example:

$$\dot{x} = -y$$
$$\dot{y} = x$$
$$\nabla \cdot f = 0$$



Example:

$\dot{x} = 1$

$\dot{y} = 1$