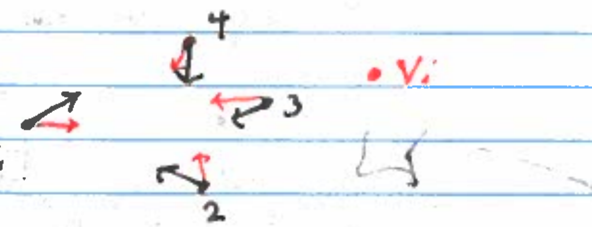
Lecture 18: Hamiltonian Systems

Classical Mechanics

$$\frac{dx_i}{dt} = v_i$$



$$m_i \frac{dv_i}{dt} = -\frac{\partial V}{\partial x_i}(x_1, \dots, x_n)$$

- The momentum is $p_i = m_i v_i$
- The configuration variables $x_i = q_i$.

Let $H(q,p) = \sum_{i=1}^{n} \frac{1}{2m_i} p_i^2 + V(q)$ (Mechanical Energy)

↖ Hamiltonian.

$$\Rightarrow \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i} = v_i = \frac{dq_i}{dt}$$

$$\frac{\partial H}{\partial q_i} = \frac{\partial V}{\partial q_i} = -m_i \frac{dv_i}{dt} = -\frac{dp_i}{dt}$$

$$\Rightarrow \boxed{\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} \end{aligned}}$$

→ 2n system of equations
n degrees of freedom

More generally:

$H: M \to \mathbb{R}$

↑ phase space of points (q,p)

q – Configuration

p – Canonical momentum.

$$\Rightarrow \frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dt}}_{\text{summation over repeated indices.}} = \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} = 0$$

Example:

2-body problem in the plane

Sun $\qquad$ Earth $\rightsquigarrow (x,y) = (q_1, q_2)$

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{\mu}{\sqrt{q_1^2+q_2^2}}$$

$$\Rightarrow \frac{dq_1}{dt} = p_1 \qquad \Bigg| \qquad \frac{dp_1}{dt} = -\frac{\mu q_1}{(q_1^2+q_2^2)^{3/2}}$$

$$\frac{dq_2}{dt} = p_2 \qquad \Bigg| \qquad \frac{dp_2}{dt} = -\frac{\mu q_2}{(q_1^2+q_2^2)^{3/2}}$$

Let $L = q_1 p_2 - q_2 p_1$

$$\frac{dL}{dt} = \frac{dq_1}{dt}p_2 + q_1\frac{dp_2}{dt} - \frac{dq_2}{dt}p_1 - q_2\frac{dp_1}{dt}$$

$$= p_1 p_2 - \frac{\mu q_1 q_2}{(q_1^2+q_2^2)^{3/2}} - p_2 p_1 + \frac{\mu q_1 q_2}{(q_1^2+q_2^2)^{3/2}}$$

$$= 0.$$

$\Rightarrow$ Angular momentum is conserved.

Observation:

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial L}{\partial p_1}\frac{dp_1}{dt} + \frac{\partial L}{\partial q_2}\frac{dq_2}{dt} + \frac{\partial L}{\partial p_2}\frac{dp_2}{dt}$$

$$= \frac{\partial L}{\partial q_1}\frac{\partial H}{\partial p_1} - \frac{\partial L}{\partial p_1}\frac{\partial H}{\partial q_1} + \frac{\partial L}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial L}{\partial p_2}\frac{\partial H}{\partial q_2}$$

$$= \frac{\partial L}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial L}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= \{L, H\}.$$

$$\boxed{\{F,H\} = \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} \quad \text{Poisson Bracket}}$$

A quantity F is conserved under the Hamiltonian flow if
$$\{F, H\} = 0.$$

Further Properties:

Let $z = (q, p)$. Then

- $\frac{dz}{dt} = J \nabla H$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \rightsquigarrow$ Poisson matrix.

- $\{F, H\} = \nabla F^T J \nabla H$

- $\dot{z} = \{z, H\}$.
  $\Rightarrow \dot{H} = \{H, H\} = 0.$

- Volume is preserved in phase space:
  $f = (\partial H / \partial p, -\frac{\partial H}{\partial q})$

  $$\Rightarrow \nabla \cdot f = \frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right) = 0.$$

  $\Rightarrow$ No asymptotically stable or neutrally stable fixed points!

- Fixed points satisfy
  $\nabla H = 0$
  $\Rightarrow$ critical points of the Hamiltonian.

Example:

Return to 2-body problem. Let:

$$r = \sqrt{q_1^2+q_2^2}, \quad \theta = \arctan(q_2/q_1), \quad p_r = \frac{q_1p_1+q_2p_2}{\sqrt{q_1^2+q_2^2}}, \quad p_\theta = q_1p_2 - q_2p_1$$

$$\Rightarrow p_r^2 + \frac{p_\theta^2}{r^2} = \frac{(q_1p_1+q_2p_2)^2}{r^2} + \frac{(q_1p_2-q_2p_1)}{r^2}$$

$$= \frac{q_1^2p_1^2 + 2q_1p_1q_2p_2 + q_2^2p_2^2 + q_1^2p_2^2 - 2q_1p_2q_2p_1 + q_2^2p_1^2}{r^2}$$

$$= \frac{(q_1^2+q_2^2)(p_1^2+p_2^2)}{r^2}$$

$$= p_1^2 + p_2^2$$

$$\Rightarrow H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r}$$

Now,

$$\dot{r} = \frac{dr}{dt} = \frac{\partial r}{\partial q_i}\frac{dq_i}{dt}$$

$$= \frac{\partial r}{\partial q_i}\frac{\partial H}{\partial p_i}$$

$$= \frac{\partial r}{\partial q_i}\left(\cancel{\frac{\partial H}{\partial r}\frac{\partial r}{\partial p_i}} + \cancel{\frac{\partial H}{\partial \theta}\frac{\partial \theta}{\partial p_i}} + \frac{\partial H}{\partial p_r}\frac{\partial p_r}{\partial p_i} + \frac{\partial H}{\partial p_\theta}\frac{\partial p_\theta}{\partial p_i}\right)$$

$$= \frac{\partial r}{\partial q_i}\left(\frac{\partial H}{\partial p_r}\frac{\partial p_r}{\partial p_i} + \frac{\partial H}{\partial p_\theta}\frac{\partial p_\theta}{\partial p_i}\right)$$

$$= \frac{q_1}{\sqrt{q_1^2+q_2^2}}\left(\frac{p_r q_1}{\sqrt{q_1^2+q_2^2}} - \frac{p_\theta q_2}{r}\right) + \frac{q_2}{\sqrt{q_1^2+q_2^2}}\left(\frac{p_r q_2}{r} + \frac{p_\theta q_1}{r}\right)$$

$$= \frac{q_1^2 p_r + q_2^2 p_r}{(q_1^2+q_2^2)}$$

$$= p_r$$

$\Rightarrow \dot{r} = \frac{\partial H}{\partial p_r}$. $\Rightarrow$ Similar Calculations show $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$, $\dot{p}_r = -\frac{\partial H}{\partial p_r}$, $\dot{p}_\theta = -\frac{\partial H}{\partial p}$

In this coordinate system the dynamics is constrained to lie on the manifold:

$$M_c = \{ (r, \theta, p_r, p_\theta) \in \mathbb{R}^+ \times S^1 \times \mathbb{R}^2 : H(r, p_r, p_\theta) = H_0, \ p_\theta = L_0 \}$$

$$= \{ (r, \theta, p_r, p_\theta) \in \mathbb{R}^+ \times S^1 \times \mathbb{R}^2 : \tfrac{1}{2}(p_r^2 + L_0^2/r^2) - \mu/r = H_0 \}$$

This manifold is non-empty

$$\frac{1}{2}\left(p_r^2 + \frac{L_\theta^2}{r^2} \right) = H_0 + \frac{\mu}{r} > 0$$