Lecture 19: Poisson Dynamics

Poisson Bracket:

Let $F, G, H \in C^2(M, \mathbb{R})$. $\{\cdot,\cdot\}: C^2(M,\mathbb{R}) \times C^2(M,\mathbb{R}) \to C^0(M,\mathbb{R})$

Satisfies:

1. Antisymmetry: $\{F, G\} = -\{G, F\}$.
2. Bilinearity: $\{F+G, H\} = \{F, H\} + \{G, H\}$, $\{aF, bG\} = ab\{F, G\}$.
3. Derivation: $\{FH, G\} = F\{H, G\} + H\{F, G\}$
4. Jacobi Identity:
$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$$

Lemma - Suppose L is a linear operator on $C^1(\mathbb{R}, \mathbb{R})$ that obeys the derivation property,
$$L(fg) = f L(g) + g L(f)$$
and $L(x) = 1$. Then $L = d/dx$.

proof:

For $n \in \mathbb{N}$ let $P(n)$ be the logical statement that $L(x^n) = nx^{n-1}$.

1. $L(x) = 1$ and thus $P(1)$ is true.
2. If $n \geq 1$ we have that if $P(X_n)$ is true then
$$\begin{aligned} L(x^{n+1}) &= L(x x^n) \\ &= x^n + x L(x^n) \\ &= x^n + n x x^{n-1} \\ &= (n+1) x^n. \end{aligned}$$

Therefore, by the principal of mathematical induction
$$L(x^n) = n x^{n-1}.$$

It follows by linearity then if p is any polynomial then
$$L(p(x)) = p'(x).$$

Now, for any $f \in C^2$ there exists a polynomial p_ε such that $\|f - p_\varepsilon\|_\infty < \varepsilon$ and $\|f' - p_\varepsilon'\| < \varepsilon$.
$$\Rightarrow \left| L(f) - \frac{df}{dx} \right| = \left| L(f-p) + L(p) - \frac{df}{dx} \right| \leq |L(f-p)| + \left|p'(x) - \frac{df}{dx}\right|$$
$$\Rightarrow \left|L(f) - \frac{df}{dx}\right| < \delta \text{, where } \delta \text{ is arbitrarily small.}$$

Lemma

Suppose $\{\cdot,\cdot\}$ is a Poisson bracket on $C^2(\mathbb{R}^d,\mathbb{R})$ then there exists antisymmetric $J(z)$ such that

$$\{F,G\} = \nabla F^T J(z) \nabla G.$$

Remark:

Unlike Hamiltonian systems J is a function of z and could be singular. Regardless, dynamics defined by

$$\dot{z} = \{z, H\} = J(z)\nabla H$$

Example:

Rigid body dynamics:

$$\omega \in \mathbb{R}^3$$

describes angular velocities about different principle axes with moments of inertia

$$I_3 > I_2 > I_1$$

$$\frac{d}{dt}L_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) L_2 L_3 \qquad H = \sum L_i^2/2I_i$$

$$\frac{d}{dt}L_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) L_3 L_1 \qquad L_i = I_i \omega_i$$

$$\frac{d}{dt}L_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) L_1 L_2$$

Poisson system with

$$J = \begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix}$$

$$\dot{z} = J \nabla H$$