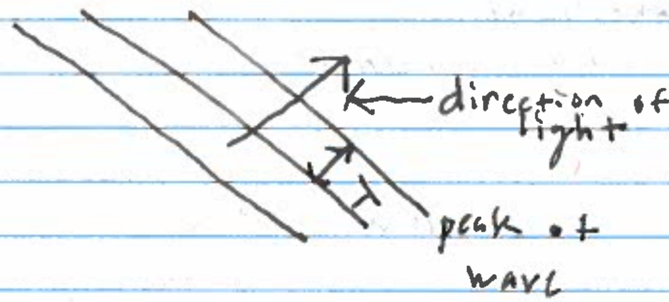
Lecture 20: The Action Principle

Method of Stationary Phase

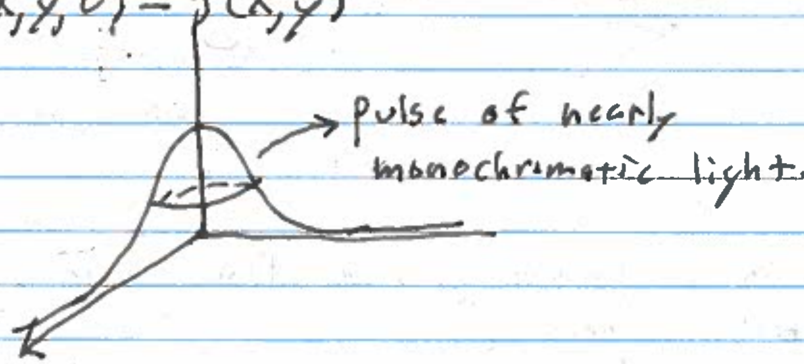
$u_{k_x,k_y}(x,y,t) = A e^{i(k_x x + k_y y + \omega t)} + \bar{A} e^{-i(k_x x + k_y y + \omega t)}$

- A ~ complex amplitude of waves
- k_x, k_y ~ wave numbers ($\lambda = 2\pi/(k_x^2+k_y^2)^{1/2}$ = wavelength)
- ω ~ angular frequency ($T = 2\pi/\omega$ = period)



$u_t = \nu \Delta u$ ⤳ probabagation of beam of light.

$u(x,y,0) = f(x,y)$



$$u(x,y,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(k_x,k_y)\, e^{-i\nu(k_x^2+k_y^2)t}\, e^{-i(k_x+k_y)t}\, dk_x\, dk_y$$

$$\hat{f}(k_x,k_y) = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(k_x+k_y)}\, dx\, dy}_{\text{superposition of waves}}$$

⇒ u ~ superposition of waves with frequency wavelength relationship.

$\omega = \nu(k_x^2+k_y^2)$.

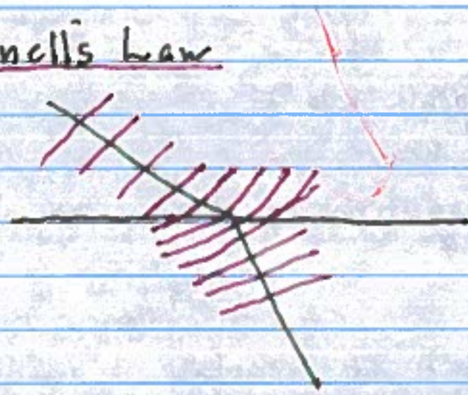
Consequence:

Huygens principle:

← light is nothing more than than a continuum of propagating waves

- observed light comes from constructive interference

Snell's Law



air ← light moves faster

water ← light moves slower

The slowing down arises from constructive interference!

How is it possible to go from wave approach to particle approach?

key idea trough an example. How to approximate
$F(\lambda)=\int_{-\pi/2}^{\pi/2}\cos(t-\lambda\cos(t))\,dt$?

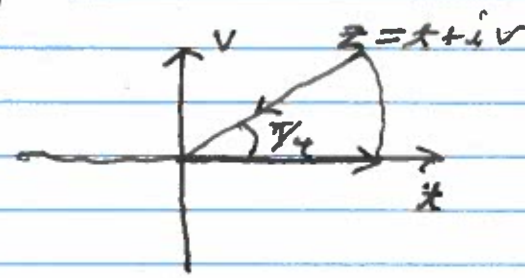
$$F(\lambda)=\mathrm{Re}\int_{-\pi/2}^{\pi/2}e^{i(t-\lambda\cos(t))}\,dt$$

$$=\mathrm{Re}\int_{-\pi/2}^{\pi/2}\underbrace{e^{it}}_{\text{high frequency waves}}\ \underbrace{e^{-i\lambda\cos(t)}}_{\text{"stationary at } t=0\text{", i.e. } \frac{d}{dt}\cos(t)=0.}\,dt$$

$$F(\lambda) \approx Re\left(\int_{-\pi/2}^{\pi/2} e^{i0} e^{-i\lambda(1-t^2/2)} dt\right)$$

$$\approx 2Re\left(\int_0^{\infty} e^{-i\lambda(1-t^2/2)} dt\right)$$

$$= 2Re\left(e^{-i\lambda}\int_0^{\infty} e^{i\lambda t^2/2} dt\right)$$



$$= 2Re\left(e^{-i\lambda}\sqrt{\frac{\pi}{2(-i\lambda)}}\right)$$

$$= \frac{1}{\sqrt{\lambda}}\left(\cos(\lambda) + \sin(\lambda)\right)$$

Hamilton's principle;

$$S[\gamma] = \int_\gamma (p\,dq - H\,dt) = \int_a^b \left(p\frac{dq}{dt} - H\right)dt = \int_a^b \left(p_i\frac{dq_i}{dt} - H\right)dt.$$

Recall:

For $f: \mathbb{R} \to \mathbb{R}$

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + o(\Delta x)$$

$$\Rightarrow \Delta f = f'(x)\,\Delta x + o(\Delta x)$$

Definition of a differential

For $f: \mathbb{R}^n \to \mathbb{R}$

$$f(x+\Delta x) = f(x) + \nabla f\,\Delta x + o(\Delta x)$$

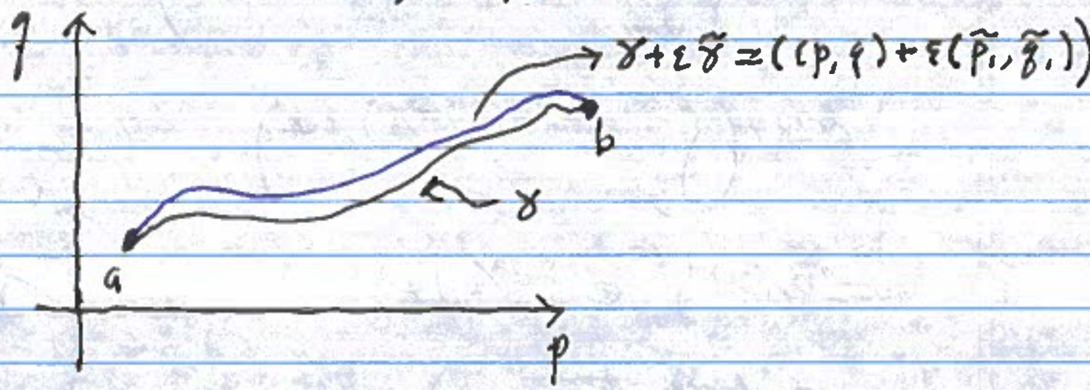
$$\Rightarrow \Delta f = \nabla f\,\Delta x + o(\Delta x)$$

operator (matrix)

$S: C^2 \to \mathbb{R}$

$$S[\gamma + \tilde{\gamma}] = S[\gamma] + L\,\delta\gamma + \dots$$

What do we mean by a perturbation?



$\tilde\gamma \in C^2(a,b)$ satisfy $\tilde\gamma(a)=\tilde\gamma(b)=0$.

$$S[\gamma+\varepsilon\tilde\gamma] = \int_a^b \left((p_i+\varepsilon\tilde p_i)\left(\frac{dq_i}{dt}+\varepsilon\frac{d\tilde q_i}{dt}\right) - H(q+\varepsilon\tilde q,\, p+\varepsilon\tilde p)\right)dt$$

Needed to quantify smallness

$$= \int_a^b \left((p_i+\varepsilon\tilde p_i)\left(\frac{dq_i}{dt}+\varepsilon\frac{d\tilde q_i}{dt}\right) - H(q,p) - \frac{\partial H}{\partial q_i}\varepsilon\tilde q_i - \frac{\partial H}{\partial p_i}\varepsilon\tilde p_i\right)dt + o(\varepsilon)$$

$$= \int_a^b \left[\left(p_i\frac{dq_i}{dt} + \varepsilon\tilde p_i\frac{dq_i}{dt} + \varepsilon p_i\frac{d\tilde q_i}{dt}\right) - H(q,p) - \frac{\partial H}{\partial q_i}\varepsilon\tilde q_i - \frac{\partial H}{\partial p_i}\varepsilon\tilde p_i\right]dt + o(\varepsilon)$$

$$= \int_a^b \left(p_i\frac{dq_i}{dt} - H(q,p)\right)dt + \int_a^b\left[\left(\frac{dq_i}{dt}-\frac{\partial H}{\partial p_i}\right)\Big|_\gamma \varepsilon\tilde p_i - \left(\frac{dp_i}{dt}+\frac{\partial H}{\partial q_i}\right)\Big|_\gamma \varepsilon\tilde q_i\right]dt$$
$$+ \cancel{\varepsilon p_i\tilde q_i}\Big|_a^b + o(\varepsilon)$$

$$\Rightarrow S[\gamma+\varepsilon\tilde\gamma] = S[\gamma] + L(\varepsilon\tilde\gamma) + o(\varepsilon),$$
$$L(\varepsilon\tilde\gamma) = \int_a^b\left[\left(\frac{dq_i}{dt}-\frac{\partial H}{\partial p_i}\right)\Big|_\gamma \varepsilon\tilde p_i - \left(\frac{dp_i}{dt}+\frac{\partial H}{\partial q_i}\right)\Big|_\gamma \varepsilon\tilde q_i\right]dt$$

Stationarity implies that (Action Principle):

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{\partial p_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Lagrangian Systems

$$A[\gamma] = \int_a^b L(\gamma(t), \dot{\gamma}(t), t)\,dt$$

Hamilton's Principle - Paths realized by the physics representing L are those for which A is stationary.

$$A[\gamma + \varepsilon\tilde{\gamma}] = \int_a^b L(\gamma + \varepsilon\tilde{\gamma}, \dot{\gamma} + \varepsilon\dot{\tilde{\gamma}}, t)\,dt$$

$$= \int_a^b \left(L(\gamma, \dot{\gamma}, t) + \frac{\partial L}{\partial \gamma}\varepsilon\tilde{\gamma} + \frac{\partial L}{\partial \dot{\gamma}}\varepsilon\dot{\tilde{\gamma}} \right) dt$$

$$= A[\gamma] + \int_a^b \left(\frac{\partial L}{\partial \gamma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} \right) \varepsilon\tilde{\gamma}\,dt$$

Dynamics satisfy:

$$\frac{\partial L}{\partial \gamma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = 0$$

Example:

$$L = \frac{1}{2}\dot{\gamma}^2 - V(\gamma)$$

$$\frac{\partial L}{\partial \gamma} = -\nabla V$$

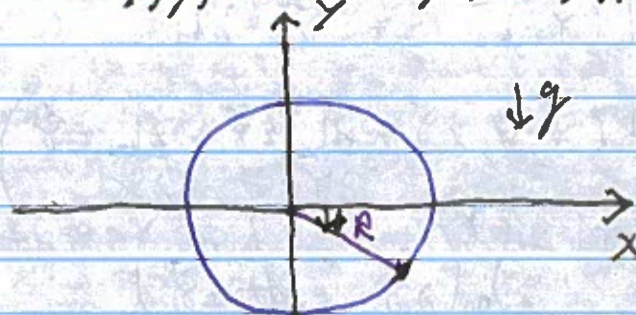
$$\frac{\partial L}{\partial \dot{\gamma}} = \dot{\gamma}$$

$$\Rightarrow -\nabla V - \ddot{\gamma} = 0$$

$$\Rightarrow \ddot{\gamma} = -\nabla V \rightarrow \text{Newton's law of motion.}$$

Theorem - Lagrangian is coordinate independent!
Suppose $h: N \to M$ is a C^2 embedding and $L(x, \dot{x}, t)$ is a Lagrangian for $x \in M$. Then the dynamics of $y \in N$ where $x = h(y)$, is given by the Euler-Lagrange equations of the Lagrangian

$$\tilde{L}(y, \dot{y}, t) = L(h(y), Dh(y), \dot{y})$$



$$L = \frac{1}{2}\dot{x}^2 + \dot{y}^2 - gy$$

$x = R\cos\theta$
$y = -R\sin\theta$
$\dot{x} = -R\sin\theta\,\dot{\theta}$
$\dot{y} = -R\cos\theta\,\dot{\theta}$

$$L = \frac{1}{2}R^2\dot{\theta}^2 + gR\sin\theta$$

$$\frac{\partial L}{\partial \theta} = gR\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = R^2\dot{\theta}$$

$$\Rightarrow \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = gR\cos\theta - R^2\ddot{\theta}$$

$$\Rightarrow \ddot{\theta} = \frac{-g}{R}\cos\theta$$