Lecture #3: Matrix ODEs

Eigenvalues and Eigenvectors

$A \in \mathbb{R}^{n\times n}$,

$A\vec{v} = \lambda \vec{v}$

eigenvector eigenvalue.

*How do you find?

$(A-\lambda I)\vec{v} = 0$

$\Rightarrow \ker(A-\lambda I) \neq \{0\}$

$\Rightarrow \det(A-\lambda I) = 0$

$\Rightarrow p(\lambda) = 0$

degree n-polynomial

Eigenvalues are roots of $p(\lambda)$.

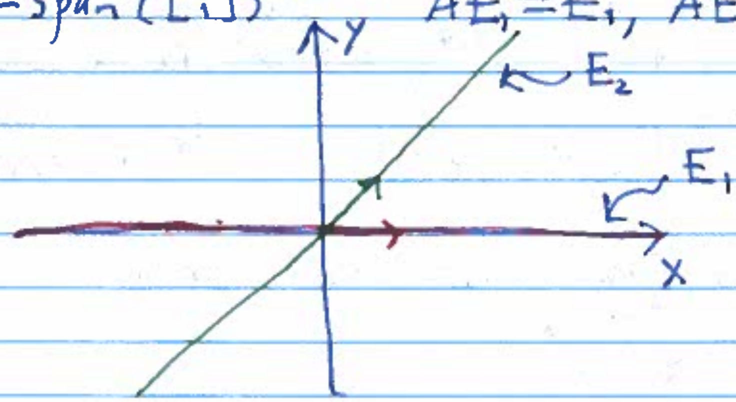
- algebraic multiplicity = multiplicity as a root
- geometric multiplicity = dimension of the span of eigenvectors associated with an eigenvalue

Example:

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\left.\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 3 \end{matrix}\right\}$ algebraic multiplicity 1.

$\left.\begin{matrix} A - I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A - 3I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{matrix}\right\}$ geometric multiplicity 1.

$E_1 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$
$E_2 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ ← Invariant subspaces since $AE_1 = E_1$, $AE_2 = E_2$

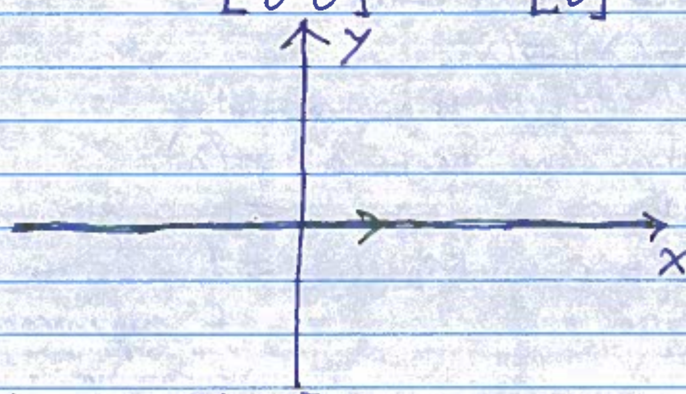


2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 1$ algebraic multiplicity of 2 and geometric multiplicity of 2

$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ (Everything is an eigenvector).

3. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 1$ algebraic multiplicity of 2

$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, geometric multiplicity of 1.



$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Linear ODEs.

Consider $\dot{\vec{x}} = A\vec{x}$, $A \in \mathbb{R}^{n \times n}$, $\vec{x} \in \mathbb{R}^n$.

1. If $\vec{x}_1, \vec{x}_2$ are solutions then so is $\vec{x}_3 = a\vec{x}_1 + b\vec{x}_2$.

proof:

$$\dot{\vec{x}}_3 = a\dot{\vec{x}}_1 + b\dot{\vec{x}}_2 = aA\vec{x}_1 + bA\vec{x}_2 = A(a\vec{x}_1 + b\vec{x}_2) = A\vec{x}_3.$$

2. If λ_i is an eigenvalue with associated eigenvector $\vec{v}_i$ then $e^{\lambda_i t}\vec{v}_i = \vec{x}_i(t)$ is a solution.

proof:

$$\dot{\vec{x}}_i(t) = \lambda_i e^{\lambda_i t}\vec{v}_i = e^{\lambda_i t} A\vec{v}_i = A\vec{x}_i.$$

3. For the initial value problem

$$\dot{\vec{x}} = A\vec{x}$$
$$\vec{x}(0) = \vec{x}_0$$

If A is full rank and is not eigenvector deficient then the unique solution is given by

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \dots + c_n e^{\lambda_n t}\vec{v}_n,$$

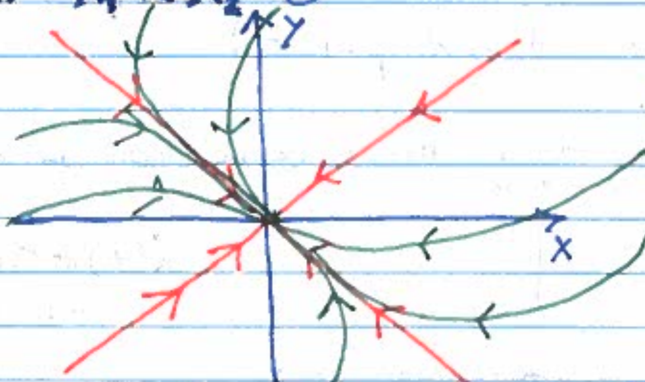
where $c_1, \dots, c_n$ satisfy:

$$\vec{x}_0 = c_1\vec{v}_1 + \dots + c_n\vec{v}_n.$$

Two Dimensional Linear ODEs.

$$\dot{\vec{x}} = A\vec{x}.$$

1. $\lambda_1 < \lambda_2 < 0$



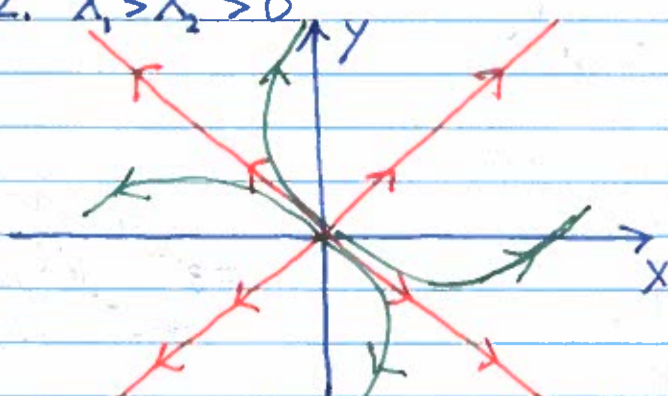
$X(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

$\Downarrow$

$e^{\lambda_1 t} \to 0$ faster than $e^{\lambda_2 t}$ thus solutions decay to E_2

Regardless, $\lim_{t\to\infty} \vec{x}(t) = 0$. (Stable Node)

2. $\lambda_1 > \lambda_2 > 0$

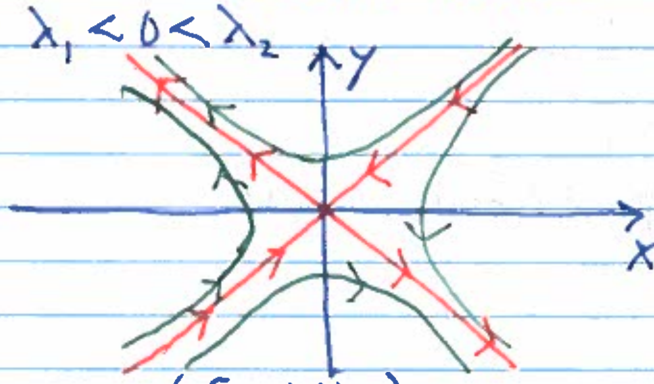


$X(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

$e^{\lambda_1 t} \to \infty$ faster than $e^{\lambda_2 t}$ and thus solutions are parallel to E_1 for large t.

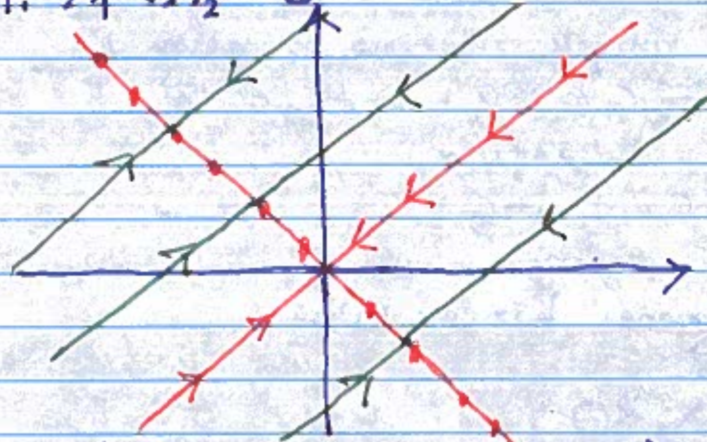
(Stable Node).

3. $\lambda_1 < 0 < \lambda_2$



(Saddle)

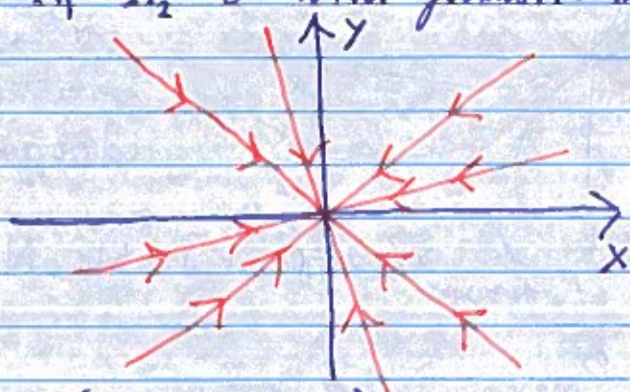
4. $\lambda_1 < \lambda_2 = 0$



$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 \vec{v}_2$.

(Line of fixed points).

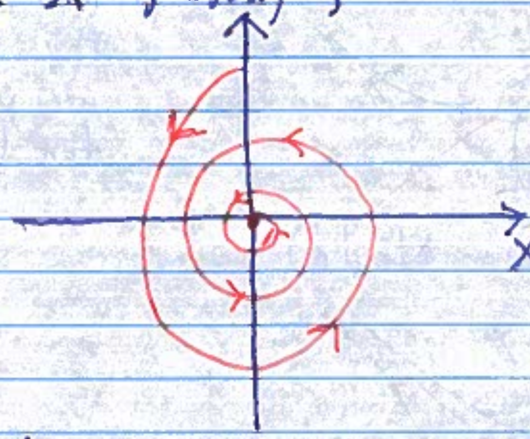
5. $\lambda_1 = \lambda_2 < 0$ with geometric multiplicity 2.



$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_1 t}\vec{v}_2 = (c_1\vec{v}_1 + c_2\vec{v}_2)e^{\lambda t}.$$

(Star Node).

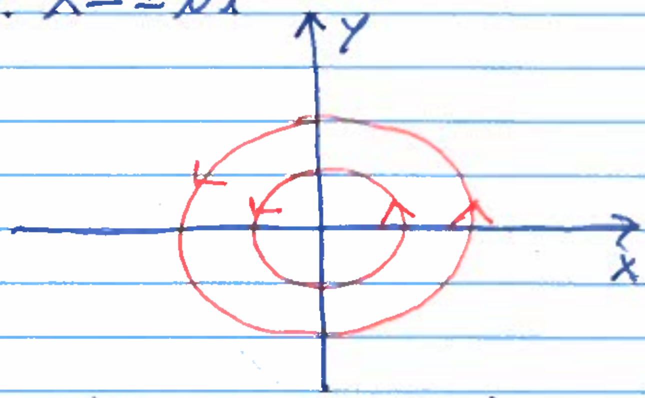
6. $\lambda = \rho \pm \omega i$, $\rho < 0$



$$\vec{x}(t) = c_1 e^{\rho t} e^{\omega i t}\vec{v}_1 + c_1^* e^{\rho t} e^{-\omega i t}\vec{v}_2^*$$
$$= e^{\rho t}\left(c_1(\cos(t) + i\sin(t))\vec{v}_1 + c_1^*(\cos(t) - i\sin(t))\vec{v}_1^*\right)$$

(Stable Spiral/Stable Focus)

7. $\lambda = \pm \nu i$



(Linear Center).

* Only case we didn't consider is an eigenvalue in which geometric multiplicity is less than algebraic multiplicity.