Lecture 4: Exponentials of Operators

Motivation:

$$\dot{\vec{x}} = A\vec{x}$$
$$\vec{x}(0) = \vec{x}_0$$

- Suppose geometric multiplicity = algebraic multiplicity.
- Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be n-linearly independent eigenvectors with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$.
- Let $P = [\vec{v}_1, \dots, \vec{v}_n]$, $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. Therefore,

$$AP = \Lambda P$$
$$\Rightarrow AP = P\Lambda$$
$$\Rightarrow A = P\Lambda P^{-1}, \quad P \sim \Lambda \text{ a diagonal matrix.}$$

If we let $\vec{y} = P^{-1}\vec{x}$ (coordinate transformation).
Then,

$$\dot{\vec{y}} = P^{-1}\frac{d\vec{x}}{dt} = P^{-1}A\vec{x} = P^{-1}AP\vec{y} = \Lambda\vec{y}.$$

$$\Rightarrow \dot{y}_1 = \lambda_1 y_1$$
$$\vdots \qquad \vdots$$
$$\dot{y}_n = \lambda_n y_n$$

$$\Rightarrow y_1 = c_1 e^{\lambda_1 t}$$
$$\vdots$$
$$y_n = c_n e^{\lambda_n t}$$

$$\boxed{\begin{aligned} &\Rightarrow \vec{y} = e^{\Lambda t}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = e^{\Lambda t}\vec{c} \\ &e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \end{aligned}}$$

Therefore,

$$\vec{x}(t;\vec{c}) = Pe^{t\Lambda}\vec{c}$$

$$\Rightarrow \vec{x}(0;\vec{c}) = P\vec{c} = \vec{x}_0$$

$$\Rightarrow \vec{x}(0;\vec{c}) = \vec{c} = P^{-1}\vec{x}_0$$

$$\Rightarrow \boxed{\vec{x}(t;\vec{x}_0) = Pe^{t\Lambda}P^{-1}\vec{x}_0}$$

Let, $\exp(tA) = Pe^{t\Lambda}P^{-1}$ then:

$$\boxed{\vec{x}(t;\vec{x}_0) = e^{tA}\vec{x}_0}$$

Functional Analysis 101:

E = normed linear space; i.e. vector space with norm $\|\cdot\|$.

- $x, y \in E \Rightarrow x+y \in E$, $ax \in E$ for all $a \in \mathbb{R}$ $(\mathbb{C})$.
- $\|ax\| = |a| \cdot \|x\|$
- $\|x+y\| \leq \|x\| + \|y\|$.
- $\|x\| = 0 \Rightarrow x = 0$.

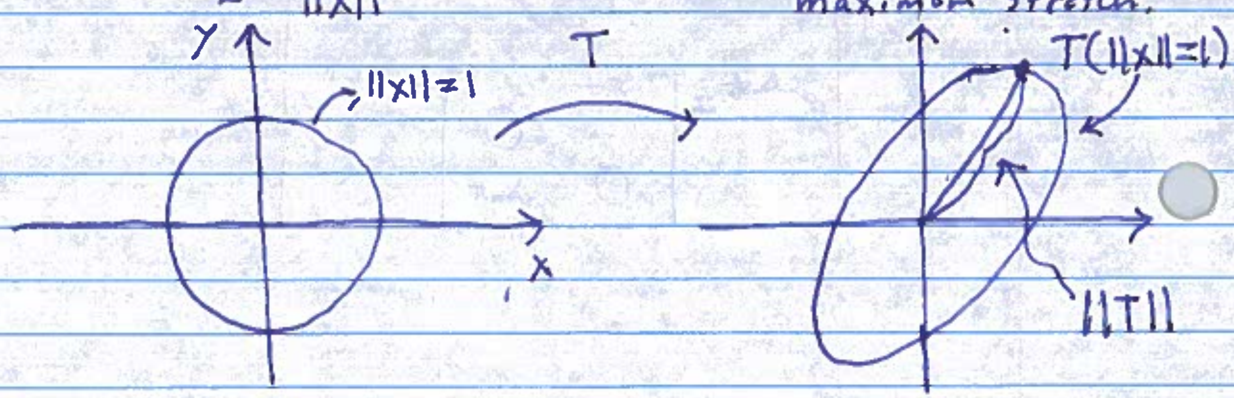
Example:

$\mathbb{R}^2$ with norm $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$.

Operator:

- $T: E_1 \to E_2$ is a (linear) operator if
  $$T(x+ay) = T(x) + aT(y).$$

- $\|T\| = \sup_{x \in E} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|$ ← Measurement of maximum stretch.



Limits:

- $x_n \in E$. $x_n \to x^* \in E$ if
$$\lim_{n\to\infty} \|x_n - x^*\| = 0$$

- $T_n \in \mathcal{L}(E_1, E_2)$, (Space of linear operators), $T_n \to T^*$ if
$$\lim_{n\to\infty} \|T_n - T\| = 0.$$

Exponential of an operator:

- $T \in \mathcal{L}(E_1, E_2)$ then
$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

Theorem -
$$\|e^T\| \leq e^{\|T\|}$$

proof:
$$\|T(x)\| \leq \|T\| \cdot \|x\|$$
$$\Rightarrow \|T^k(x)\| = \|T(T^{k-1}(x))\|$$
$$\leq \|T\| \cdot \|T^{k-1}(x)\|$$
$$\vdots$$
$$\leq \|T\|^k \|x\|.$$

$$\Rightarrow \left\| \sum_{n=0}^{M} \frac{1}{n!} T^n(x) \right\| \leq \sum_{n=0}^{M} \frac{1}{n!} \|T\|^n \cdot \|x\| \leq e^{\|T\|} \cdot \|x\|$$

$$\Rightarrow \left\| \sum_{n=0}^{\infty} \frac{1}{n!} T^n(x) \right\| \leq e^{\|T\|} \|x\|$$

$$\Rightarrow \|e^T(\bar{x})\| \leq e^{\|T\|} \|x\|$$

$$\Rightarrow \frac{\|e^T(\bar{x})\|}{\|x\|} \leq e^{\|T\|}.$$

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Properties:

1. $e^0 = I$
2. $(e^A)^{-1} = e^{-A}$

proof:

$$e^{-A}e^{A} = (I - A + \tfrac{1}{2}A^2 - \tfrac{1}{6}A^3 + \ldots)(I + A + \tfrac{1}{2}A^2 + \tfrac{1}{6}A^3 + \ldots)$$
$$= I + A - A + \tfrac{1}{2}A^2 + \tfrac{1}{2}A^2 - A^2 + \ldots)$$
$$= I.$$

3. If $[AB] = AB - BA = 0$ then
$$e^A e^B = e^{A+B}.$$

proof:

$$e^A e^B = (I + A + \tfrac{1}{2}A^2 + \ldots)(I + B + \tfrac{1}{2}B^2 + \ldots)$$
$$= I + (A+B) + \tfrac{1}{2}(A^2 + 2AB + B^2) + \ldots$$
$$= I + (A+B) + \tfrac{1}{2}(A^2 + AB + BA + B^2) + \ldots$$
$$= e^{A+B}.$$

4. $e^{BAB^{-1}} = Be^A B^{-1}$

proof:

$$e^{BAB^{-1}} = I + (BAB^{-1}) + \frac{1}{2}(BAB^{-1})^2 + \ldots$$
$$= I + BAB^{-1} + \frac{1}{2}BA^2B^{-1} + \ldots$$
$$= B(I + A + \tfrac{1}{2}A^2 + \ldots)B^{-1}$$
$$= Be^A B^{-1}$$

5. If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then
$$e^A\vec{v} = e^{\lambda}\vec{v}$$

proof:

$$e^A\vec{v} = (I + A + \tfrac{1}{2}A^2 + \ldots)\vec{v}$$
$$= (\vec{v} + \lambda\vec{v} + \tfrac{1}{2}\lambda^2\vec{v} + \ldots)$$
$$= (1 + \lambda + \tfrac{1}{2}\lambda^2 + \ldots)\vec{v}$$
$$= e^{\lambda}\vec{v}.$$

Fundamental Solution Theorem

Theorem - Let $A \in \mathbb{R}^{n \times n}$. The initial value problem

$$\dot{\vec{x}} = A\vec{x}$$
$$\vec{x}(0) = \vec{x}_0$$

has the unique solution

$$x(t) = e^{At}\vec{x}_0.$$

~~proof~~:

- $$\frac{d}{dt} e^{tA} = \lim_{h \to 0} \frac{e^{(t+h)A} - e^{tA}}{h}$$
$$= \lim_{h \to 0} \frac{(e^{hA} - I)e^{tA}}{h}$$
$$= \lim_{h \to 0} \frac{(hA + h^2/2 A^2 + \dots)e^{tA}}{h}$$
$$= \lim_{h \to 0} (A + h/2 A^2 + \dots)e^{tA}$$
$$= Ae^{tA}.$$

Therefore,

$$\frac{d}{dt} e^{At}\vec{x}_0 = Ae^{At}\vec{x}_0.$$

- Now suppose y is another solution.

$$\frac{d}{dt}\left(e^{-tA}\vec{y}(t)\right) = -Ae^{-tA}\vec{y}(t) + e^{-tA}A\vec{y}(t) = 0$$

$$\Rightarrow e^{-tA}\vec{y}(t) = \vec{y}_0$$
$$\Rightarrow \vec{y}(t) = e^{tA}\vec{y}_0.$$