# Lecture 5: Complex and Repeated Eigenvalues

## Complex Eigenvalues:

$A \in \mathbb{R}^{n\times n}$

1. If $\lambda = a + ib$ is an eigenvalue then $\lambda^* = v - iw$ is also an eigenvalue.

~~proof~~:

The operation $*$ satisfies $(\lambda^n)^* = (\lambda^*)^n$ thus for any polynomial

$$p(\lambda^*) = p(\lambda)^*.$$

2. If $\vec{v}$ is an eigenvector so is $\vec{v}^*$.

"Diagonalization by rotations"

Suppose $\lambda = a + ib$, $\vec{v} = \vec{v} + i\vec{w}$

$$\Rightarrow A\vec{v} = \lambda\vec{v}$$

$$\Rightarrow A(\vec{v} + i\vec{w}) = (a + ib)(\vec{v} + i\vec{w})$$

$$\Rightarrow \begin{matrix} A\vec{v} = a\vec{v} - b\vec{w} \\ A\vec{w} = b\vec{v} + a\vec{w} \end{matrix} \Rightarrow \text{span}\{\vec{v}, \vec{w}\} \text{ is invariant.}$$

$$\Rightarrow A[\vec{v}|\vec{w}] = [\vec{v}|\vec{w}]\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

## Exponential of "Rotation":

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

$$= a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + b\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\sigma \rightsquigarrow \text{rotation}}$$

$$\Rightarrow [aI, b\sigma] = ab\sigma - ab\sigma = 0$$

$$\Rightarrow \exp(A) = \exp(aI)\exp(b\sigma)$$

Note:

$$\sigma^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \quad (\sigma \sim i)$$

$$\Rightarrow \sigma^3 = -\sigma, \quad \sigma^4 = I, \ldots$$

$$\Rightarrow \exp(b\sigma) = I + b\sigma - \frac{b^2 I}{2} - \frac{b^3\sigma}{3!} + \cdots$$

$$= \begin{bmatrix} 1 - b^2/2 + \cdots & 0 \\ 0 & 1 - b^2/2 \end{bmatrix} + \begin{bmatrix} 0 & -b + b^3/3! + \cdots \\ b - b^3/3! + \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(b) & 0 \\ 0 & \cos(b) \end{bmatrix} + \begin{bmatrix} 0 & -\sin(b) \\ \sin(b) & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$$

Therefore,

$$\exp(A) = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix} \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$$

$$= e^a \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}.$$

## Application to ODEs

$$\dot{\vec{x}} = A\vec{x}$$

If $A$ is diagonalizable with $k$ real eigenvalues and $2\ell$ complex eigenvalues $a_j + ib_j$ then

$$\vec{x}(t) = e^{tA}\vec{x}_0$$

$$A = P \begin{bmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & \lambda_k & & & & \\ & & & & a_1 & -b_1 & & \\ & 0 & & & b_1 & a_1 & & \\ & & & & & & \ddots & \\ & & & & & & & \begin{bmatrix} a_\ell & -b_\ell \\ b_\ell & a_\ell \end{bmatrix} \end{bmatrix} P^{-1}$$

## Multiple Eigenvalues:

- If $\lambda_j$ is an eigenvalue of $A$ with algebraic multiplicity $n_j$: the generalized eigenspace of $\lambda_j$ is:

$$E_j = NS[(A-\lambda_j I)^{n_j}]$$

- A space $E$ is invariant under $T$ if for all $v \in E$, $T(v) \in E$.

- $E_j$ is invariant.

~~proof~~:

Suppose $v \in E_j$

$$\Rightarrow (A-\lambda_j I)^{n_j} v = 0$$

Now,

$$(A-\lambda_j I)^{n_j} A v = A(A-\lambda_j I)^{n_j} v = 0.$$

- A generalized eigenvector satisfies

$$(T-\lambda_j I)^{n_j} v_j = 0$$

where $n_j$ is the algebraic multiplicity.

## Jordan Canonical Form -

$$\vec{x}(t) = e^{tA}\vec{x}_0$$

$$A = P\begin{bmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & 0 & & \\ & & \lambda_k & & & & & & \\ & & & a_1 & -b_1 & & & & \\ & & & b_1 & a_1 & & & & \\ & 0 & & & & \ddots & & & \\ & & & & & & a_l & -b_l & \\ & & & & & & b_l & a_l & \\ & & & & & & & & \ddots \\ & & & & & & & & \lambda_{k+1} \; 1 \\ & & & & & & & & \quad \lambda_{k+1} \; \ddots \end{bmatrix}$$

Jordan Block.

Suppose $J = \lambda I + N$,

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\lambda I, J] = 0$$

$$\Rightarrow J^2 = (\lambda I + N)^2$$
$$= \lambda^2 I + 2\lambda N + N^2$$
$$= \begin{bmatrix} \lambda^2 & 2\lambda & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$$

$$J^3 = (\lambda I + N)^3$$
$$= \lambda^3 I + 3\lambda^2 N + 3\lambda N^2$$

$$J^4 = (\lambda I + N)^4 = \lambda^4 I + 4\lambda^3 N + 6\lambda^2 N^2$$

$$\vdots$$

$$\Rightarrow \exp(tJ) = I + tJ + \frac{t^2 J^2}{2} + \frac{t^3 J^3}{6} + \ldots$$

$$= \begin{bmatrix} 1 + t\lambda + t^2\lambda^2/2 + \ldots & t + 2\lambda t^2/2 + 3\lambda^2 t^3/6 + \ldots & t^2/2 + 3\lambda t^3/6 \ldots \\ 0 & 1 + t\lambda + t^2\lambda^2/2 + \ldots & t + 2\lambda t^2/2 + 3\lambda^2 t^3/6 \ldots \\ 0 & 0 & 1 + t\lambda + t^2\lambda^2/2 + \ldots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & t(1 + t\lambda + \tfrac{1}{2} t^2\lambda^2 + \ldots) & t^2/2\,(1 + \lambda t + \ldots) \\ 0 & e^{\lambda t} & t(1 + t\lambda + \tfrac{1}{2} t^2 \lambda^2 + \ldots) \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} & t^2/2\, e^{\lambda t} \\ 0 & e^{\lambda t} & t e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$