Lecture 8: Solving Equations.

Idea:

Assuming f is continuous solve:

$$f(x) = x.$$

Make a (good) guess x_0 and iterate

$$x_1 = f(x_0),\ x_2 = f(x_1),\ \ldots,\ x_{n+1} = f(x_n).$$

If $x_n \to x^*$ then

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(x^*).$$

$\Rightarrow x^*$ solves the equation (Brilliant), also proves existence!!

ODEs:

$$\dot{x} = f(x,t)$$
$$x(0) = x_0$$

$$\Rightarrow \int_0^t \dot{x}\,dt = \int_0^t f(x(t),t)\,dt$$

$$\boxed{x(t) = x_0 + \int_0^t f(x(t),t)\,dt}$$

This is an integral equation

$$x(t) = I[x(t)]$$

where

$$I[x(t)] = x_0 + \int_0^t f(x(t),t)\,dt.$$

Can we make a (good) guess and iterate??

$$x_1 = I[x_0],\ x_2 = I[x_1],\ \ldots,\ x_{n+1} = I[x_n].$$

Example:

$$\dot{x}(t) = -tx$$
$$x(0) = x_0$$

Exact solution is given by $x(t) = x_0 e^{-t^2/2}$. If we guess

$$x_0(t) = x_0$$

$$\Rightarrow x(t) = x_0 + \int_0^t -t x_0\,dt = x_0(1 - t^2/2)$$

$$\Rightarrow x_2(t) = x_0 + \int_0^t -x_0(1 - t^2/2)\, t\, dt$$
$$= x_0(1 - t^2/2 + t^4/8)\, dt$$

$$\Rightarrow x_3(t) = x_0 + \int_0^t -x_0(1 - t^2/2 + t^4/8)\, t\, dt$$
$$= x_0\,(1 - t^2/2 + t^4/8 - t^6/48)$$

Note, $e^{-t^2/2} = 1 - t^2/2 + \frac{1}{2}\, t^4/4 - \frac{1}{6}\, t^6/8 + \ldots$

Example:

$$\dot{x} = x(1-x)$$
$$x(0) = x_0$$
$$\Rightarrow x(t) = \frac{x_0 e^t}{1 - x_0 + x_0 e^t}$$

Iterative solution.

$$x(t) = x_0 + \int_0^t x(1-x)\, dt$$
$$x_1(t) = x_0 + \int_0^t x_0(1-x_0)\, dt$$

$$\Rightarrow x_1(t) = x_0 + x_0(1-x_0)\, t$$

$$x_2(t) = x_0 + \int_0^t (x_0 t + x_0(1-x_0) t^2)\, dt$$
$$= x_0 + \frac{x_0 t^2}{2} + \frac{x_0(1-x_0) t^3}{3}$$

Questions:

- Does the sequence of functions converge?
  (Need notion of convergence)
- Does sequence of functions converge to solution?
  (Need notion of continuity)

Fixed Points For Functions. (1922!!)

Simplest proof.

$f: \mathbb{R} \to \mathbb{R}$

satisfies, $|f'| < M < 1$. Consider the equation $f(x) = x$. This equation has a unique solution.

proof:

Consider the sequence $x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots$

Assume $m \geq n \geq 1$ then

$$
\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} + \ldots + x_{n+1} - x_n| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \ldots + |x_{n+1} - x_n| \\
&= |f(x_{m-1}) - f(x_{m-2})| + |f(x_{m-2}) - f(x_{m-3})| + \ldots + |f(x_{n}) - x_n| \\
&= f'(c_1)\,|x_{m-1} - x_{m-2}| + |f'(c_2)|\,|x_{m-2} - x_{m-3}| + \ldots + |x_{n+1} - x_n| \\
&\leq M|x_{m-1} - x_{m-2}| + M|x_{m-2} - x_{m-3}| + \ldots + |x_{n+1} - x_n| \\
&\ \ \vdots \\
&\leq (M^{(m-n)} + M^{(m-n-1)} + \ldots + M + 1)\,|x_{n+1} - x_n| \\
&\leq (M^{(m-n)} + M^{(m-n-1)} + \ldots + M + 1)\,M\,|x_n - x_{n-1}| \\
&\ \ \vdots \\
&\leq (M^{(m-n)} + M^{(m-n-1)} + \ldots + M + 1)\,M^n\,|x_1 - x_0| \\
&= (M^m + M^{m-1} + \ldots + M^{n+1} + M^n)\,|x_1 - x_0|
\end{aligned}
$$

Since, $\lim_{n\to\infty} (M^m + M^{m-1} + \ldots + M^{n+1} + M^n) = 0$ assuming $m \geq n$ it follows that x_n is Cauchy. Therefore, there exists x^* such that $\lim_{n\to\infty} x_n = x^*$. Furthermore, by continuity

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(x^*).$$

Suppose x^*, y^* are both solutions. Therefore,

$$|x^* - y^*| = |f(x^*) - f(y^*)|$$