Lecture 9: Function Spaces

Vector Fields

$E \subset \mathbb{R}^n$

$f: E \to \mathbb{R}^n$

$\Downarrow$

$f(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n))$

$\Rightarrow f(\vec{x}) = (f_1(\vec{x}),\dots,f_n(\vec{x}))$.

$f_i : E \to \mathbb{R}$.

We say f_i is continuous if for all $\vec{x}_n \in E$ satisfying

$x_n \to x^*$, i.e, $\lim_{n\to\infty} \|\vec{x}^* - x_n\| = 0$

it follows that

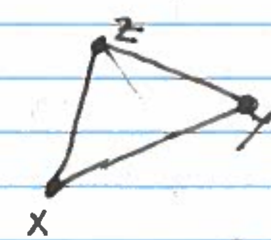
$\lim_{n\to\infty} f(x_n) = f(x^*)$, i.e, $\lim_{n\to\infty} |f(x_n) - f(x^*)| = 0$.

$*\, C^0(E;\mathbb{R}^n) = \{$ functions $f: E \to \mathbb{R}^n$, such that each component is continuous $\}$.

Metric Spaces/Normed Linear Spaces

(X, ρ) is a metric space with metric ρ if

1. For all $x, y \in X$, $\rho(x,y) \geq 0$ and $\rho(x,y) = 0$ if and only if $x = y$.
2. For all $x, y \in X$,
   $\rho(x,y) = \rho(y,x)$.
3. For all $x, y, z \in X$,
   $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$



For all norms $\|\cdot\|$, the associated metric is defined by:

$\|x - y\| = \rho(x,y)$.

L^p spaces:

$\mathbb{X} = C^0(E; \mathbb{R}^n)$

Normed linear spaces

$(\mathbb{X}, \|\cdot\|_p), \quad p \geq 1$

with

$$\boxed{\begin{aligned} \|f\|_p &= \int_E (|f_1(\vec{x})|^2 + \dots + |f_n(\vec{x})|^2 \, d\vec{x})^{1/p} \\ &= \left(\int_E \|f(\vec{x})\|_2^p \, d\vec{x}\right)^{1/p}. \\ \|f\|_\infty &= \sup_{x \in E} \{(|f_1(\vec{x})|^2 + \dots + |f_n(\vec{x})|^2)^{1/2}\} \\ &= \sup_{x \in E} \|f(\vec{x})\|_2 \end{aligned}}$$

- We say $f_n \in C^0(E; \mathbb{R}^n)$ converges to f^* w.r.t. L^p convergence if
$$\lim_{n \to \infty} \|f_n - f^*\|_p = 0$$

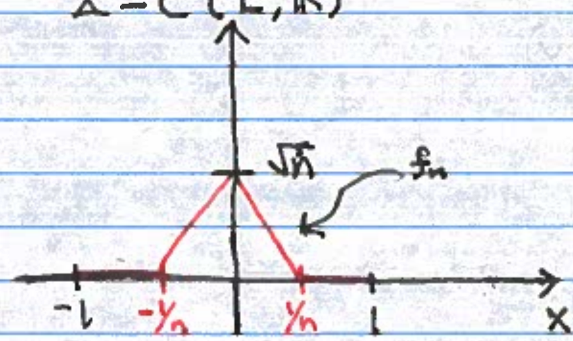
- We say $f_n \in C^0(E; \mathbb{R}^n)$ is Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies
$$\|f_n - f_m\| < \varepsilon.$$

- $(\mathbb{X}, \|\cdot\|)$ is complete if all Cauchy sequences converge to an element of $\mathbb{X}$.

Examples of Strangeness:

$E = [-1, 1]$

$\mathbb{X} = C^0(E; \mathbb{R})$



- $\|f_n\|_1 = \int_{-1}^{1} |f_n(x)| \, dx = 2/\sqrt{n}$!

$\Rightarrow \lim_{n \to \infty} \|f_n\|_1 = 0$

$\Rightarrow f_n \xrightarrow{L^1} 0$

- $\|f_n\|_2 = \left(\int_{-1}^{1} |f_n(x)|^2 \, dx\right)^{1/2} = \sqrt{2}$!

$\Rightarrow f_n \overset{L^2}{\not\to} 0$

- $\|f_n\|_\infty = \sqrt{n} \Rightarrow \lim_{n \to \infty} \|f_n\|_\infty = \infty$!

L^∞-Norm

<u>Theorem</u> - If $f_n \in C^0(E;\mathbb{R}^n)$ satisfies $f_n \xrightarrow{L^\infty} f$ then $f_n \xrightarrow{L^p} f$ for all $p < \infty$.

proof:

$$\|f_n - f\|_{L^p} = \left(\int_E \|f_n(x) - f(x)\|_2^p \, dx\right)^{1/p}$$

$$\leq \left(\int_E \|f_n - f\|_\infty^p \, dx\right)^{1/p}$$

$$= \|f_n - f\|_\infty \cdot \mathrm{vol}(E).$$

Therefore, by squeeze theorem

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0.$$

<u>Theorem</u> - The space $C(E;\mathbb{R}^n, \|\cdot\|_\infty)$ is complete when E is compact.

proof:

Suppose $f_n \in C^0(E;\mathbb{R}^n, \|\cdot\|_\infty)$ is a Cauchy sequence. Therefore, for all $x \in E$ and $m,n \in \mathbb{N}$ it follows that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$$

and thus as a sequence of numbers $f_n(x)$ is Cauchy and thus has a limit. Define $f^*: E \to \mathbb{R}^n$ by

$$f^*(x) = \lim_{n\to\infty} f_n(x).$$

Since f_n is Cauchy, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m,n \geq N$ implies

$$\|f_m - f_n\|_\infty < \varepsilon$$

$$\Rightarrow |f_m(x_0) - f_n(x_0)| < \varepsilon$$

Therefore, since $|\cdot|: \mathbb{R} \to \mathbb{R}$ is continuous it follows that

$$|f^*(x_0) - f_n(x_0)| = \lim_{m\to\infty} |f_m(x_0) - f_n(x_0)| < \varepsilon$$

Therefore,

$$\sup_{x_0 \in E} |f^*(x_0) - f_n(x_0)| = \|f^* - f_n\|_\infty < \varepsilon$$

and thus $\lim_{n\to\infty} \|f^* - f_n\|_\infty = 0.$

Furthermore, for $x, y \in E$ it follows that

$$|f^*(x) - f^*(y)| = |f^*(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f^*(y)|$$

$$\leq |f^*(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f^*(y)|$$

Since f is continuous and $f_n(x) \to f(x)$ and $f_n(y) \to f(y)$ for $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $\delta(n)$ such that

$$|f^*(x) - f^*(y)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

and thus f^* is continuous. ∎