MTH 357/657
Homework #10

Due Date: April 14, 2023

1 Covariance

1. The joint probability distribution $p(x, y)$ of random random variables X and Y satisfies

$$\begin{aligned}
&p(0,0) = \frac{1}{12},\ p(1,0) = \frac{1}{6},\ p(2,0) = \frac{1}{24},\\
&p(0,1) = \frac{1}{4},\ p(1,1) = \frac{1}{4},\ p(2,1) = \frac{1}{40},\\
&p(0,2) = \frac{1}{8},\ p(1,2) = \frac{1}{20},\\
&p(0,3) = \frac{1}{120}.
\end{aligned}$$

   Find $\mathrm{Cov}(X, Y)$.

2. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

   find $\mathrm{Cov}(X, Y)$.

3. Suppose X, Y are discrete random variables with joint probability distribution $p(x, y)$ satisfying $p(-1, 1) = 1/4$, $p(0, 0) = 1/6$, $p(1, 0) = 1/12$, $p(1, 1) = 1/2$ and is zero for all other values. Show that

   (a) $\mathrm{Cov}(X, Y) = 0$

   (b) The two random variables are not independent.

4. Suppose the probability density of X is given by

$$f(x) = \begin{cases} 1 + x & -1 < x \leq 0 \\ 1 - x & 0 < x < 1 \\ - & \text{elsewhere} \end{cases}$$

   and $U = X$ and $V = X^2$. Show that

   (a) $\mathrm{Cov}(U, V) = 0$

   (b) U and V are dependent.

5. If X_1, X_2, X_3 are independent and have the means 4, 9, and 3 and the variances 3, 7, and 5, find the mean and the variance of

   (a) $Y = 2X_1 - 3X_2 + 4X_3$,

   (b) $Z = X_1 + 2X_2 - X_3$.

6. If the joint probability density of X and Y is given by

$$p(x,y) = \begin{cases} \frac{1}{3}(x+y) & \text{for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

   find the variance of $W = 3X + 4Y - 5$.

7. A quarter is bent so that probabilities of heads and tails are .40 and .60. If is tossed twice, what is the covariance of Z, the number of heads obtained on the first toss, and W the total number of heads obtained in the two tosses of the coin?

8. The inside diameter of a cylindrical tube is a random variable with a mean of 3 inches and a standard deviation of .02 inch, the thickness of the tube is a random variable with a mean of .3 inch and a standard deviation of .005 inch, and the two random variables are independent. Find the mean and the standard deviation of the outside diameter of the tube.

2 Conditional Expectation

1. With reference to problem #1 from the "Covariance" section, find the conditional mean and the conditional variance of X given $Y = 1$.

2. With reference to problem #2 from the "Covariance" section, find the conditional mean and the conditional variance of Y given $X = 1/4$.

Homework #10

#2

If the joint probability density of X and Y is given by

$$p(x,y)=\begin{cases}\frac{1}{4}(2x+y), & 0<x<1,\ 0<y<2\\ 0, & \text{elsewhere}\end{cases}$$

find $Cov(X,Y)$.

Solution:

Calculating, we have that

- $$\mathbb{E}[X]=\int_0^1\int_0^2 \frac{1}{4}(2x^2+xy)\,dy\,dx$$
$$=\frac{1}{4}\int_0^1 (2x^2y+\frac{1}{2}xy^2)\Big|_0^2 dx$$
$$=\frac{1}{4}\int_0^1 (4x^2+2x)\,dx$$
$$=\frac{1}{4}(\tfrac{4}{3}x^3+x^2)\Big|_0^1$$
$$=\frac{1}{4}(\tfrac{4}{3}+1)$$
$$=\tfrac{1}{4}(\tfrac{7}{3})$$
$$=\tfrac{7}{12}.$$

- $$\mathbb{E}[Y]=\frac{1}{4}\int_0^1\int_0^2 (2xy+y^2)\,dy\,dx$$
$$=\frac{1}{4}\int_0^1 (xy^2+\frac{1}{3}y^3)\Big|_0^2 dx$$
$$=\frac{1}{4}\int_0^1 (4x+\frac{8}{3})\,dx$$
$$=\frac{1}{4}(2x^2+\frac{8}{3}x)\Big|_0^1$$
$$=\frac{1}{4}(2+\tfrac{8}{3})$$
$$=\tfrac{14}{12}$$
$$=\tfrac{7}{6}$$

- $$\mathbb{E}[XY]=\frac{1}{4}\int_0^1\int_0^2 (2x^2y+xy^2)\,dy\,dx$$
$$=\frac{1}{4}\int_0^1 (x^2y^2+\frac{1}{3}xy^3)\Big|_0^2 dx$$
$$=\frac{1}{4}\int_0^1 (4x^2+\tfrac{8}{3}x)\,dx$$
$$=\frac{1}{4}\int_0^1 (\frac{4}{3}x^3+\tfrac{4}{3}x^2)\Big|_0^1$$
$$=\tfrac{2}{3}.$$

Therefore,

$$\begin{aligned} Cov(X,Y) &= 2/3 - 7/12 \cdot 7/6 \\ &= 2/3 - 49/72 \\ &= 48/72 - 49/72 \\ &= -1/72. \end{aligned}$$

$\blacksquare$

#3.

Suppose X, Y are discrete random variables with joint probability distribution $p(x,y)$ satisfying $p(-1,1) = 1/4$, $p(0,0) = 1/6$, $p(1,0) = 1/12$, $p(1,1) = 1/2$ and is zero for all other values. Show that

(a) $Cov(X,Y) = 0$

(b) X, Y are not independent.

<u>Solution:</u>

(a) Calculating we have that

- $\mathbb{E}[X] = -1 \cdot 1/4 + 1/12 + 1/2 = -3/12 + 1/12 + 6/12 = 1/3$
- $\mathbb{E}[Y] = 1 \cdot 1/4 + 1 \cdot 1/2 = 3/4$
- $\mathbb{E}[XY] = -1 \cdot 1/4 + 1 \cdot 1/2 = 1/4$

Therefore, $Cov(X,Y) = 1/4 - 1/3 \cdot 3/4 = 0.$

(b) $p(1,1) = 1/4 = P(X=1, Y=1)$. However,

$$P(X=1) = 1/12 + 1/2 = 7/12$$

$$P(Y=1) = 1/4 + 1/2 = 3/4$$

Therefore,

$$P(X=1, Y=1) \neq P(X=1)P(Y=1).$$

and thus X, Y are not independent.

$\blacksquare$

#4.

Suppose the probability of X is given by

$$p(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

and $U = X$ and $V = X^2$. Show that

(a) $Cov(U, V) = 0$

(b) U and V are dependent.

Solution:

(a). Since $\mathbb{E}[X] = 0$ we have that

$$\begin{aligned} Cov(U,V) &= Cov(X, X^2) \\ &= \mathbb{E}[X(X^2 - \mathbb{E}[X^2])] \\ &= \mathbb{E}[X^3] - \mathbb{E}[X\mathbb{E}[X^2]] \\ &= \mathbb{E}[X^3] - \mathbb{E}[X^2]\mathbb{E}[X] \\ &= \mathbb{E}[X^3]. \end{aligned}$$

Now, since $p(x)$ is even it follows that $x^3 p(x)$ is odd and thus $\mathbb{E}[X^3] = 0$ proving that $Cov(U, V) = 0$.

(b). $$\begin{aligned} P(0 < U < 1/2, 0 < V < 1/4) &= P(0 < X < 1/2, 0 < X^2 < 1/4) \\ &= P(0 < X < 1/2). \end{aligned}$$

Therefore, if U, V are independent it follows that

$$P(0 < X < 1/2)P(0 < X^2 < 1/4) = P(0 < X < 1/2)$$

$$\Rightarrow P(0 < X^2 < 1/4) = 1,$$

$$\Rightarrow P(-1/2 < X < 1/2) = 1,$$

$$\Rightarrow 2\int_0^{1/2} (1-x)\,dx = 1$$

$$\Rightarrow 2\left(x - \tfrac{1}{2}x^2\right)\Big|_0^{1/2} = 1$$

$$\Rightarrow 1 - 1/4 = 1$$

which is a contradiction. ∎

#6.

If the joint probability density of X and Y is given by

$$p(x,y)=\begin{cases} \frac{1}{3}(x+y), & 0<x\le 1,\ 0<y<2 \\ 0, & \text{elsewhere} \end{cases}$$

find the variance of $W=3X+4Y-5$.

Solution:

Since for any random variable Z and constant C,

$$\mathrm{Cov}(Z,C)=\mathbb{E}[(Z-\mu_Z)(C-C)]=0$$

we have that

$$\begin{aligned}\mathrm{Cov}(W,W)&=\mathrm{Cov}(3X+4Y-5,\ 3X+4Y-5)\\ &=\mathrm{Cov}(3X+4Y,\ 3X+4Y)\\ &=9\,\mathrm{Cov}(X,X)+24\,\mathrm{Cov}(X,Y)+16\,\mathrm{Cov}(Y,Y)\end{aligned}$$

Now,

$$\begin{aligned}\mathbb{E}[X]&=\int_0^1\int_0^2 \tfrac{1}{3}(x^2+xy)\,dy\,dx\\ &=\tfrac{1}{3}\int_0^1 (x^2y+\tfrac{1}{2}xy^2)\Big|_0^2\,dx\\ &=\tfrac{1}{3}\int_0^1 (2x^2+2x)\,dx\\ &=\tfrac{1}{3}(\tfrac{2}{3}x^3+x^2)\Big|_0^1\\ &=\tfrac{1}{3}(\tfrac{2}{3}+1)\\ &=\tfrac{5}{9}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Y]&=\tfrac{1}{3}\int_0^1\int_0^2 (xy+y^2)\,dy\,dx\\ &=\tfrac{1}{3}\int_0^1 (\tfrac{1}{2}xy^2+\tfrac{1}{3}y^3)\Big|_0^2\,dx\\ &=\tfrac{1}{3}\int_0^1 (\tfrac{1}{2}x\cdot 4+\tfrac{8}{3})\,dx\\ &=\tfrac{1}{3}(x^2+\tfrac{8}{3}x)\Big|_0^1\\ &=\tfrac{1}{3}(1+\tfrac{8}{3})\\ &=\tfrac{11}{9}.\end{aligned}$$

Lets compute the marginal densities to save time!

$$f(x) = \int_0^2 1/3\,(x+y)\,dy$$
$$= 1/3\,(xy + \tfrac{1}{2}y^2)\Big|_0^2$$
$$= 1/3\,(2x+2)$$

$$g(y) = 1/3 \int_0^1 (x+y)\,dx$$
$$= 1/3\,(\tfrac{1}{2}x^2 + xy)\Big|_0^1$$
$$= 1/3\,(\tfrac{1}{2} + y)$$

Therefore,

$$\mathbb{E}[X^2] = \int_0^1 1/3\,(2x^3 + 2x^4)\,dx$$
$$= 1/3\,(1/2\,x^4 + 2/3\,x^3)\Big|_0^1$$
$$= 1/3\,(1/2 + 2/3)$$
$$= 7/18$$

$$\mathbb{E}[Y^2] = \int_0^2 1/3\,(\tfrac{1}{2}y^2 + y^3)\,dy$$
$$= 1/3\,(1/6\,y^3 + 1/4\,y^4)\Big|_0^2$$
$$= 1/3\,(8/6 + 4)$$
$$= 1/3\,(32/6)$$
$$= 32/18$$

$$\mathbb{E}[XY] = \int_0^1 \int_0^2 \tfrac{1}{3}(x^2y + xy^2)\,dy\,dx$$
$$= 1/3 \int_0^1 (\tfrac{1}{2}x^2y^2 + \tfrac{1}{3}xy^3)\Big|_0^2\,dx$$
$$= 1/3 \int_0^1 (2x^2 + 8/3\,x)\,dx$$
$$= 1/3\,(2/3 + 4/3)$$
$$= 6/9$$
$$= 2/3$$

Consequently,

$$\mathrm{Cov}(X,X) = 7/18 - 25/81 = 13/162$$
$$\mathrm{Cov}(X,Y) = 2/3 - 55/81 = -1/81$$
$$\mathrm{Cov}(Y,Y) = 32/18 - 121/81 = 23/81$$

Therefore,

$$\mathrm{Cov}(W,W) = 9 \cdot 13/162 + 24(-1/81) + 16 \cdot 23/81 = 805/162. \blacksquare$$

#7

A quarter is bent so that the probabilities of heads and tails are .40 and .60. If it is tossed twice, what is the covariance of Z, the number of heads obtained in the first toss, and W the total number of heads obtained in two tosses of the coin?

Solution:

It is convient to express the joint distribution in a table.

$W \backslash Z$	0	1
0	.36	0
1	.24	.24
2	0	.16

and thus

$$\mathbb{E}[Z] = .24 + .16 = .4$$

$$\mathbb{E}[W] = .48 + 2 \cdot .16 = .8$$

$$\mathbb{E}[ZW] = (.24) + 2 \cdot .16 = .56.$$

Therefore,

$$\mathrm{Cov}(Z, W) = .56 - .32 = .24$$