MTH 357/657
Homework #11

Due Date: April 21, 2023

1 Distribution Function Technique

1. If the probability density of X is given by

$$f(x) = \begin{cases} 2xe^{-x^2} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and $Y = X^2$, find

(a) the cumulative distribution function of Y,

(b) the probability density of Y.

2. If X has an exponential distribution with parameter $\beta > 0$, i.e. a gamma distribution with $\alpha = 1$, use the distribution function technique to find the probability density of $Y = \ln(X)$.

3. If X has a uniform density over the interval $[0, 1]$, use the distribution function technique to find the probability density of the random variable $Y = \sqrt{X}$.

4. Suppose X_1 and X_2 are independent random variables having exponential densities with parameters β_1 and β_2 and $Y = X_1 + X_2$.

(a) Use the distribution technique to find the probability density of Y if $\beta_1 \neq \beta_2$.

(b) Use the distribution technique to find the probability density of Y if $\beta_1 = \beta_2$.

(c) Show that if $\beta_1 = \beta_2 = 1$, the random variable

$$Z = \frac{X_1}{X_1 + X_2}$$

has a uniform density over the interval $[0, 1]$.

5. Let X be the amount of gasoline (in 1,000 gallons) that a service station has in its tanks at the beginning of the day, and Y the amount that the service station sells during that day. If the joint density of X and Y is given by

$$p(x, y) = \begin{cases} \frac{1}{200} & \text{for } 0 < y < x < 20 \\ 0 & \text{elsewhere} \end{cases}$$

use the distribution function technique to find the probability density of the amount that the service station has left in its tank at the end of the day.

6. The percentages of copper and iron in a certain kind of ore are, respectively, X and X. The joint density of these two random variables is given by

$$p(x,y) = \begin{cases} \frac{3}{11}(5x+y) & x>0,\ y>0,\ x+2y<2 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) Use the distribution function technique to find the probability density of $Z = X + Y$.

   (b) Also, find $\mathbb{E}[Z]$, the expected total percentage of copper and iron in the ore.

2 Transformation Technique

1. If X has a binomial distribution with $n = 3$ trials and $p = 1/3$, find the probability distributions of

   (a) $Y = \frac{X}{1+X}$.

   (b) $U = (X-1)^4$.

2. If $X = \ln(Y)$ has a normal distribution with mean μ and standard deviation σ, Y is said to have a **log-normal distribution**.

   (a) Find the probability density of Y

   (b) Show that the log-normal distribution has a relative maximum at $e^{\mu-\sigma^2}$.

3. If the probability density of X is given by

$$f(x) = \begin{cases} \dfrac{kx^3}{(1+2x)^6} & \text{for } x>0 \\ 0 & \text{elsewhere} \end{cases}$$

   where k is an appropriate constant to be found later.

   (a) Find the probability density of $Y = \dfrac{2X}{1+2X}$.

   (b) Identify the distribution of Y, and thus determine the value of k.

4. If X has a uniform density on the interval $[0,1]$, show that the random variable $Y = -2\ln(X)$ has a gamma distribution.

5. According to the Maxwell-Boltzmann law from statistical mechanics, the probability density of V, the speed of a gas molecule, is

$$p(v) = \begin{cases} kv^2e^{-\beta v^2} & \text{for } v>0 \\ 0 & \text{elsewhere} \end{cases},$$

   where β depends on its mass m and the absolute temperature and k is an appropriate constant.

   (a) Determine the normalization constant as a function of β.

   (b) Show that the kinetic energy $E = \dfrac{1}{2}mV^2$ is a random variable with a gamma distribution.

Homework #11

#2.

If X has an exponential distribution with parameter $\beta > 0$, use the distribution function technique to find the probability density of $Y = \ln(X)$.

Solution:

Let
$$F(y) = P(Y \leq y) = P(\ln(X) \leq y) = P(X \leq \exp(y)) = \begin{cases} \int_0^{e^y} \frac{1}{\beta} e^{-x/\beta} dx, & y \geq 0 \\ 0, & y \leq 0 \end{cases}$$

Therefore, the density is given by
$$f(y) = \frac{dF}{dy} = \begin{cases} 1/\beta\, e^{y} e^{-e^{y}/\beta}, & y \geq 0 \\ 0, & y \leq 0 \end{cases}$$
$$\Rightarrow f(y) = \begin{cases} 1/\beta\, e^{y + e^{y}/\beta}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

#4

Suppose X_1 and X_2 are independent random variables with parameters β_1 and β_2 and $Y = X_1 + X_2$.

(a) Use the distribution technique to find the probability density of Y if $\beta_1 \neq \beta_2$.

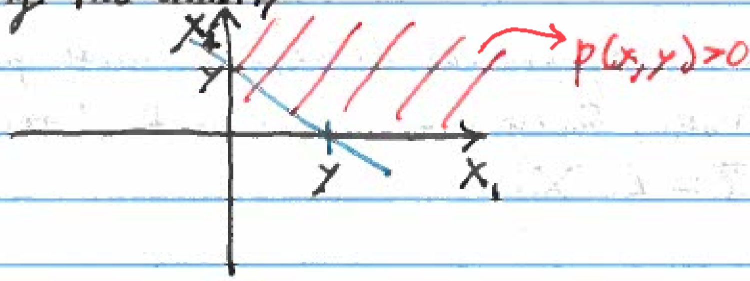
(b) Use the distribution technique to find the probability density of Y if $\beta_1 = \beta_2$

(c) Show that if $\beta_1 = \beta_2 = 1$, the random variable
$$Z = X_1/(X_1 + X_2)$$
has a uniform density on the interval $[0, 1]$.

Solution:

(a) $F(y) = P(Y \le y) = P(X_1 + X_2 \le y) = P(X_2 \le y - X_1)$.

Sketching the density



it follows that

$$P(X_2 \le y - X_1) = \begin{cases} \int_0^y \int_0^{y-x_1} \frac{1}{\beta_1\beta_2} e^{-x_1/\beta_1} e^{-x_2/\beta_2}\, dx_2\, dx_1, & y>0 \\ 0, & y \le 0 \end{cases}$$

Now,

$$\beta_1^{-1}\beta_2^{-1}\int_0^y\int_0^{y-x_1} e^{-x_1/\beta_1} e^{-x_2/\beta_2}\, dx_2\, dx_1 = \beta_1^{-1}\int_0^y e^{-x_1/\beta_1}\left(1-e^{-(y-x_1)/\beta_2}\right)dx_1$$

$$= \beta_1^{-1}\left(\int_0^y e^{-x_1/\beta_1} - e^{-y/\beta_2} e^{-x_1\cdot(\beta_2-\beta_1)/\beta_1\beta_2}\, dx_1\right)$$

$$= 1 - e^{-y/\beta_1} - \frac{e^{-y/\beta_2}}{\beta_1}\int_0^y e^{-x_1(\beta_2-\beta_1)/\beta_1\beta_2}\, dx_1$$

$$= 1 - e^{-y/\beta_1} - \frac{e^{-y/\beta_2}}{\beta_1}\,\frac{\beta_1\beta_2}{\beta_2-\beta_1}\left(1-e^{-y(\beta_2-\beta_1)/\beta_1\beta_2}\right)$$

$$= 1 - e^{-y/\beta_1} - \frac{e^{-y/\beta_2}\beta_2}{\beta_2-\beta_1} + \frac{e^{-y/\beta_1}\beta_2}{\beta_2-\beta_1}$$

$$= 1 + \frac{e^{-y/\beta_1}\beta_1}{\beta_2-\beta_1} - \frac{e^{-y/\beta_2}\beta_2}{\beta_2-\beta_1}$$

$$= 1 + \frac{\beta_1 e^{-y/\beta_1} - e^{-y/\beta_2}\beta_2}{\beta_2-\beta_1}$$

Therefore,

$$f(y) = \frac{dF}{dy} = \frac{e^{-y/\beta_2} - e^{y/\beta_1}}{\beta_2-\beta_1}$$

(b) Returning to the calculation for the case when $\beta_1 = \beta_2$ we have that

$$F(y) = 1 - e^{-y/\beta_1} - \frac{e^{-y/\beta_2}}{\beta_1}\int_0^y e^{-x_1(\beta_2-\beta_1)/\beta_1\beta_2}\,dx_1$$

$$= 1 - e^{-y/\beta_1} - \frac{e^{-y/\beta_1}}{\beta_1}\int_0^y dx_1$$

$$= 1 - e^{-y/\beta_1} - \frac{ye^{-y/\beta_1}}{\beta_1}$$

Therefore,

$$f(y) = \frac{dF}{dy} = \frac{1}{\beta_1}e^{-y/\beta_1} - \frac{1}{\beta_1}e^{-y/\beta_1} + \frac{ye^{-y/\beta_1}}{\beta_1^2}$$

$$\Rightarrow f(y) = \frac{ye^{-y/\beta_1}}{\beta_1^2}$$

(c) If we let $Z = \frac{X_1}{X_1+X_2}$ and assume $\beta_1 = \beta_2 = 1$ then

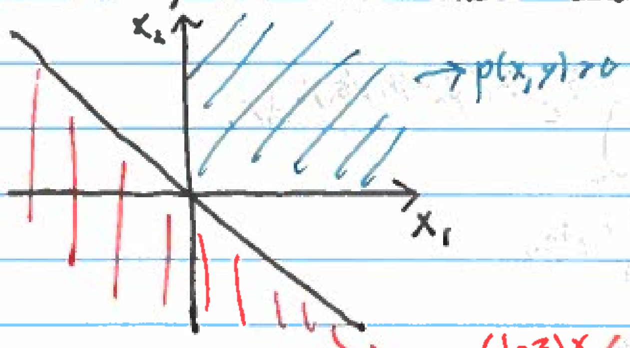
$$G(z) = P(Z \leq z)$$

$$= P\left(\frac{X_1}{X_1+X_2} \leq z\right)$$

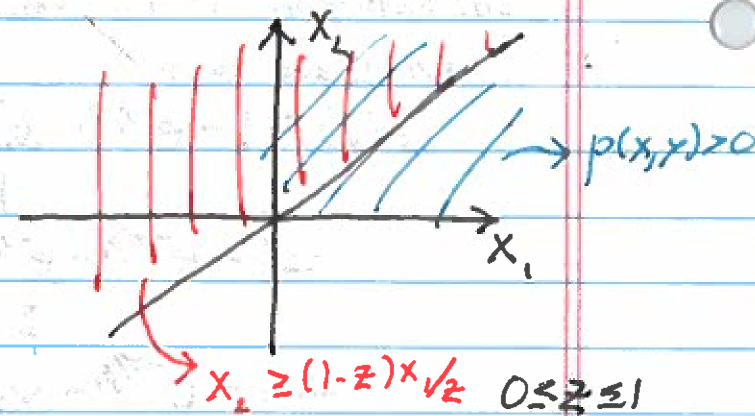
$$= P(X_1 \leq zX_1 + zX_2)$$

$$= \begin{cases} P\left(X_2 \geq \frac{(1-z)X_1}{z}\right) & \text{if } z > 0 \\ P\left(X_2 \leq \frac{(1-z)X_1}{z}\right) & \text{if } z < 0. \end{cases}$$

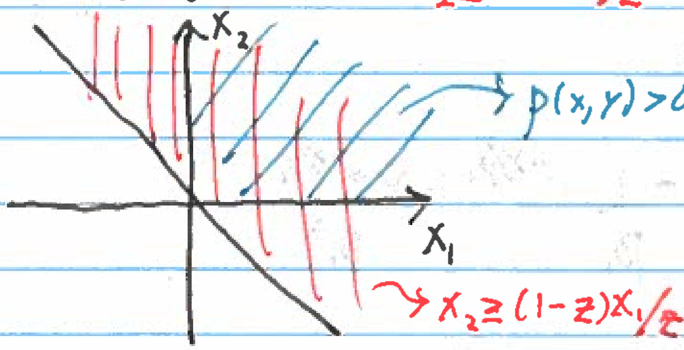
Therefore, we have three cases.



$z<0$



$0 \le z \le 1$



$z>1$

Therefore,

$$G(z) = \begin{cases} 0, & z \le 0 \\ \int_0^\infty \int_{(1-z)x_1/z}^{\infty} e^{-x_1} e^{-x_2}\, dx_2\, dx_1, & 0<z<1 \\ 1, & z \ge 1 \end{cases}$$

$$= \begin{cases} 0, & z \le 0 \\ \int_0^\infty e^{-x_1} e^{-(1-z)x_1/z}\, dx_1, & 0<z<1 \\ 1, & z \ge 1 \end{cases}$$

$$= \begin{cases} 0, & z \le 0 \\ \int_0^\infty e^{-x_1/z}\, dx_1 & 0<z<1 \\ 1, & z \ge 1 \end{cases}$$

$$= \begin{cases} 0, & z \le 0 \\ z, \; 0 \le z \le 1, & 0<z<1 \\ 1, & z \ge 1 \end{cases}$$

Therefore, the probability density is given by

$$g(z) = \frac{dG}{dz} = \begin{cases} 1, & 0 < z = 1 \\ 0, & \text{elsewhere} \end{cases}$$

#6

If X has a binomial distribution with $n=3$ trials and $p=1/3$, find the probability distributions of

(a) $Y = X/1+X$

(b) $U = (X-1)^4$

Solution:

(a) The probability distribution of X is given by

$$p(x) = \binom{3}{x}(1/3)^x(2/3)^{3-x}, \quad x = 0, 1, 2, 3$$

Since $X = Y/1-Y$ it follows that

$$g(y) = \binom{3}{y/1-y}\left(\frac{1}{3}\right)^{y/1-y}\left(\frac{2}{3}\right)^{3-y/1-y}$$

for $y = 0, 1/2, 2/3, 3/4$. As a table we have

x	0	1	2	3
p(x)	8/27	12/27	6/27	1/27

$\Rightarrow$

y	0	1/2	2/3	3/4
g(y)	8/27	12/27	6/27	1/27

(b) I am only going to write the table. The values U can take on are $U = 1, 0, 16$. Moreover,

$$P(U=0) = P(X=1) = 12/27$$

$$P(U=1) = P(X=0 \text{ or } X=2) = 8/27 + 6/27 = 14/27$$

$$P(U=16) = P(X=3) = 1/27$$

u	0	1	16
f(u)	12/27	14/27	1/27

#3

If the probability density of X is given by

$$f(x)=\begin{cases} kx^3/(1+2x)^6, & x>0 \\ 0, & \text{elsewhere} \end{cases},$$

where k is an appropriate constant to be found later.

(a) Find the probability density of $Y = 2X/1+2X$

(b) Identify the distribution of Y, and thus determine the value of k.

Solution:

(a) Assuming $a,b>0$ we have that

$$P(a<Y<b) = P(a< 2X/1+2X < b)$$
$$= P(a(1+2x)<2X<b(1+2x))$$
$$= P\left(\frac{a}{2(1-a)} < x < \frac{b}{2(1-b)}\right).$$

Assuming $a,b\in(0,1)$ we have that

$$P(a<Y<b)=\int_{a/2(1-a)}^{b/2(1-b)} \frac{kx^3}{(1+2x)^6}\,dx.$$

Letting $y = 2x/1+2x$ we have that

$$dy = \frac{(1+2x)2-2x\cdot 2}{(1+2x)^2}\,dx$$
$$= \frac{2}{(1+2x)^2}\,dx$$

and thus

$$P(a<Y<b)=\int_a^b \frac{kx^3}{2(1+2x)^4}\,dy$$
$$=\int_a^b \frac{k}{16}\,\frac{y^3}{1+2x}\,dy$$

$$\Rightarrow P(a<Y<b)=\int_a^b \frac{K}{16}\,\frac{y^3}{1+2y/2(1-y)}\,dy$$

$$=\int_a^b \frac{K}{32}(1-y)y^3\,dy.$$

Therefore, the density of Y is a beta function:

$$g(y)=\begin{cases} K/32\,(1-y)y^3, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

(b) Since this density is a beta function we have that

$$\frac{K}{32}=\frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}=\frac{6!}{3!}=6\cdot5\cdot4$$

$$\Rightarrow K=32\cdot6\cdot5\cdot4$$

$$\Rightarrow K=3840.$$

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#5

According to the Maxwell-Boltzmann law from statistical mechanics, the probability density of V, the speed of a gas molecule, is

$$p(v)=\begin{cases} kv^2e^{-\beta v^2}, & v>0 \\ 0, & \text{elsewhere} \end{cases}.$$

(a) Determine the normalization constant K.

(b) Show that the kinetic energy $E=\frac{1}{2}mV^2$ is a random variable with a gamma distribution.

Solution:

(a) $\int_0^\infty k v^2 e^{-\beta v^2} dv = \int_0^\infty k/2\beta \, v e^{-u} du = \int_0^\infty k/2\beta \, 1/\beta^{1/2} \, u^{1/2} e^{-u} du = \frac{k}{2\beta^{3/2}} \Gamma(3/2)$

Therefore,

$$K = \frac{2\beta^{3/2}}{\Gamma(3/2)} = \frac{4\beta^{3/2}}{\sqrt{\pi}}.$$

(b) $P(a<E<b) = P(a < \frac{1}{2} m V^2 < b)$

$$= P\left(\sqrt{2a/m} < V < \sqrt{2b/m}\right)$$

$$= \int_{\sqrt{2a/m}}^{\sqrt{2b/m}} k v^2 e^{-\beta v^2} dv$$

Letting $E = \frac{1}{2} m V^2$ and thus $dE = m V dV$. giving us

$$P(a<E<b) = \int_a^b k/m \, v e^{-2\beta/m E} dE$$

$$= \int_a^b k/m \sqrt{2/m} \, E e^{-2\beta/m E} dE$$

Therefore, E has a gamma density with

$$g(E) = \frac{\sqrt{2} k}{m^{3/2}} E e^{-2\beta/m E}.$$