MTH 357/657
Homework #6

Due Date: March 02, 2023

1 Moment Generating Functions

2 1. Given that X has the probability distribution $f(x) = \frac{1}{8}\binom{3}{x}$ for $x = 0, 1, 2$, and 3, find the moment-generating function of this random variable and use it to determine μ_1' and μ_2'.

2. Find μ, μ_2', and σ^2 for the random variable X that has the probability distribution $f(x) = 1/2$ for $x = -2$ and $x = 2$.

2 3. If the random variable X has the mean μ and the standard deviation σ, show that the random variable
$$Z = \frac{X - \mu}{\sigma}$$
satisfies
$$\mathbb{E}(Z) = 0 \text{ and } \mathbb{E}\left(Z^2\right) = 1.$$

2 4. The symmetry or skewness (lack of symmetry) of a distribution is often measured by means of the quantity
$$\alpha_3 = \frac{\mu_3}{\sigma^3}.$$
Draw histograms and calculate α_3 for probability distributions $f(x)$ and $g(x)$ satisfying

(a) $f(1) = .05$, $f(2) = .15$, $f(3) = .30$, $f(4) = .30$, $f(5) = .15$, and $f(6) = .05$;
(b) $g(1) = .05$, $g(2) = .20$, $g(3) = .15$, $g(4) = .45$, $g(5) = .10$, and $g(6) = .05$.

The first distribution is symmetrical while the second has a "tail" on the left-hand side and is said to be negatively skewed.

The extent to which a distribution is peaked or flat, also called the kurtosis of the distribution, is often measured by means of the quantity
$$\alpha_4 = \frac{\mu_4}{\sigma^4}.$$
Draw histograms and calculate α_4 for probability distributions $f(x)$ and $g(x)$ satisfying

(a) $f(-3) = .06$, $f(-2) = .09$, $f(-1) = .10$, $f(0) = .5$, $f(1) = .10$, $f(2) = .09$, and $f(3) = .06$.
(b) $f(-3) = .04$, $f(-2) - .11$, $f(-1) = .20$, $f(0) = .30$, $f(1) = .20$, $f(2) = .11$, and $f(3) = .04$.

2 6. Find the moment generating function of the discrete random variable X that has the probability distribution
$$f(x) = 2\left(\frac{1}{3}\right)^x \text{ for } x = 1, 2, 3, \ldots$$
and use it to determine the values of μ_1' and μ_2'.

2 Tchebysheff's Theorem

2 1. What is the smallest value of k in Tchebysheff's theorem for which the probability that random variable will take on a value between $\mu - k\sigma$ and $\mu + k\sigma$ is

   (a) at least .95;

   (b) at least .99.

2. If we let $k\sigma = c$ in Tchebysheff's theorem, what does this theorem assert about the probability that a random variable will take on a value between $\mu - c$ and $\mu + c$.

3. The number of marriage licenses issued in a certain city during the month of June may be looked upon as a random variable with $\mu = 124$ and $\sigma = 7.5$. According to Tchebysheff's theorem, with what probability can we assert that between 64 and 184 marriage licenses will be issued during the month of June.

Homework #6

#1.

Given that $f(x) = \frac{1}{8}\binom{3}{x}$ for $x = 0, 1, 2,$ and 3, find the moment generating function of this random variable and use it to determine μ_1' and μ_2'.

Solution:

$$m(t) = \mathbb{E}[e^{tX}] = \tfrac{1}{8}\left(\binom{3}{0}e^0 + \binom{3}{1}e^t + \binom{3}{2}e^{2t} + \binom{3}{3}e^{3t}\right)$$
$$= \tfrac{1}{8}(1 + 3e^t + 3e^{2t} + e^{3t})$$
$$= \tfrac{1}{8}(1+e^t)^3.$$

Therefore,

$$m'(t) = \tfrac{3}{8}(1+e^t)^2 e^t$$
$$m''(t) = \tfrac{6}{8}(1+e^t)e^{2t} + \tfrac{3}{8}(1+e^t)^2 e^t$$

and thus

$$m'(0) = \mu_1' = 12/8 = 3/2$$
$$m''(0) = \mu_2' = 6/8 \cdot 2 + 3/8 \cdot 4 = 24/8 = 3.$$

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#3.

If the random variable X has the mean μ and standard deviation σ, show that the random variable

$$Z = \frac{X - \mu}{\sigma}$$

Satistics

$$\mathbb{E}[Z] = 0 \text{ and } \mathbb{E}[Z^2] = 1.$$

Solution:

$$\mathbb{E}[Z] = \mathbb{E}[(X-\mu)/\sigma]$$
$$= \frac{1}{\sigma}(\mathbb{E}[X] - \mathbb{E}[\mu])$$
$$= \frac{1}{\sigma}(\mu - \mu) = 0.$$
$$\mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \mathbb{E}[Z^2]$$
$$= \mathbb{E}[(X-\mu)^2/\sigma^2]$$
$$= \frac{1}{\sigma^2}\mathbb{E}[(X-\mu)^2]$$
$$= \frac{1}{\sigma^2}\sigma^2$$
$$= 1.$$

■

#4.

The symmetry of skewness of a distribution is often measured by means of the quantity

$$\alpha_3 = \frac{\mu_3}{\sigma^3}.$$

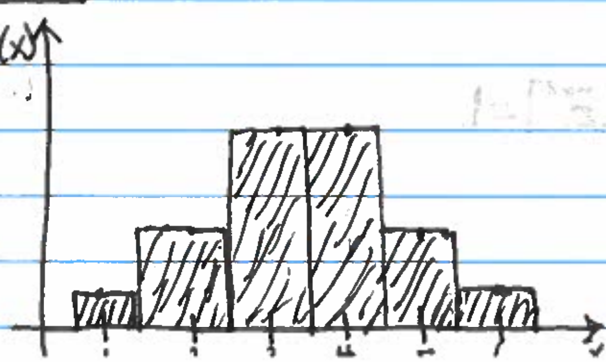
Draw histograms and calculate α_3 for probability distributions $f(x)$ and $g(x)$ satisfying

(a) $f(1)=.05$, $f(2)=.15$, $f(3)=.30$, $f(4)=.30$, $f(5)=.15$ and $f(6)=.05$;

(b) $g(1)=.05$, $g(2)=.20$, $g(3)=.15$, $g(4)=.45$, $g(5)=.10$ and $g(6)=.05$.

Solution:

(a)



Calculating, we have that

$$\mu_1' = .05 + 2\cdot.15 + 3\cdot.30 + 4\cdot.30 + 5\cdot.15 + 6\cdot.05 = 3.5.$$

$$\mu_2' = .05 + 4\cdot.15 + 9\cdot.30 + 16\cdot.30 + 25\cdot.15 + 36\cdot.05 = 13.7.$$

Thus $\sigma = (\mu_2' - \mu_1'^2)^{1/2} = 1.20416$. Therefore,

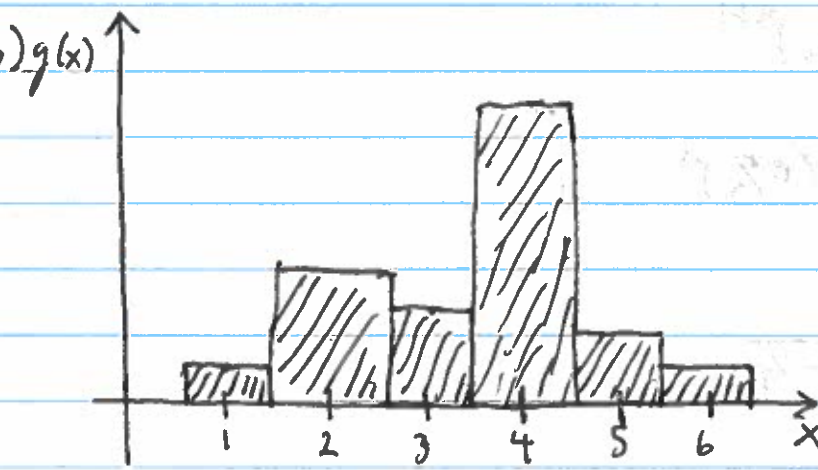
$$\alpha_3 = \frac{E[(X-\mu_1')^3]}{\sigma^3}$$

$$= [(1-3.5)^3\cdot.05 + (2-3.5)^3\cdot.15 + (3-3.5)^3\cdot.30 + (4-3.5)^3\cdot.30 + (5-3.5)^3\cdot.15 + (6-3.5)^3\cdot.05]/(1.20416)^3$$

$$= [(-2.5)^3\cdot.05 + (-1.5)^3\cdot.15 + (-.5)^3\cdot.30 + (.5)^3\cdot.30 + (1.5)^3\cdot.15 + (2.5)^3\cdot.05)]/(1.20416)^3$$

$$= 0.$$

(b)



Calculating, we have that

$$\mu_1' = .05 + 2\cdot.20 + 3\cdot.15 + 4\cdot.45 + 5\cdot.10 + 6\cdot.05 = 3.5$$

$$\mu_2' = .05 + 4\cdot.20 + 9\cdot.15 + 16\cdot.45 + 25\cdot.10 + 36\cdot.05 = 13.7$$

Thus $\sigma = (\mu_2' - \mu_1'^2)^{1/2} = 1.20416$. Furthermore,

$$E[(X-\mu_1')^3] = -2.5^3\cdot.05 - 1.5^3\cdot.20 - .5^3\cdot.15 + .5^3\cdot.45 + 1.5^3\cdot.10 + 2.5^3\cdot.05$$

$$= -.10\cdot1.5^3 + .5^3\cdot.30$$

$$= -.3$$

Therefore,

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = \frac{-.3}{1.20416^3} = -.1718.$$

#6

Find the moment generating function of the discrete random variable X that has the probability distribution

$$f(x) = 2\left(\tfrac{1}{3}\right)^x \quad \text{for } x = 1, 2, 3$$

and use it to determine μ_1' and μ_2'.

<u>Solution:</u>

$$m(t) = E\left[e^{tX}\right]$$
$$= \sum_{x=1}^{\infty} e^{tx} \cdot 2\left(\tfrac{1}{3}\right)^x$$
$$= 2\sum_{x=1}^{\infty} \left(e^t/3\right)^x$$
$$= 2\sum_{x=0}^{\infty} \left(e^t/3\right)^{x+1}$$
$$= \frac{2e^t}{3} \sum_{x=0}^{\infty} \left(e^t/3\right)^x$$
$$= \frac{2e^t}{3} \, \frac{1}{1 - e^t/3}$$
$$= \frac{2e^t}{3 - e^t}.$$

Consequently,

$$m'(t) = \frac{(3-e^t)2e^t + 2e^{2t}}{(3-e^t)^2}$$
$$= \frac{6e^t}{(3-e^t)^2}$$
$$m'(t) = \frac{(3-e^t)^2 6e^t - 6e^t \cdot 2(3-e^t)(-e^t)}{(3-e^t)^4}$$

$$\Rightarrow \mu_1' = \frac{6}{4} = \frac{3}{2}, \quad \mu_2' = \frac{4\cdot 6 + 12\cdot 2}{16} = \frac{24}{16} = \frac{3}{2}.$$

#1

What is the smallest value of K in Tchebysheff's theorem for which the random variable will take on a value between $\mu - K\sigma$ and $\mu + K\sigma$ is

(a) at least .95;

(b) at least .99.

Solution:

(a) Since $P(|X - \mu| \geq K\sigma) \leq 1/K^2$ we need

$$1/K^2 = .05 = 5/100$$

$$\Rightarrow K^2 = 100/5$$

$$\Rightarrow K = 10/\sqrt{5} = 4.47.$$

(b) Likewise we need

$$1/K^2 = .01 = 1/100$$

$$\Rightarrow K = 10.$$

$\blacksquare$