MTH 357/657
Homework #6

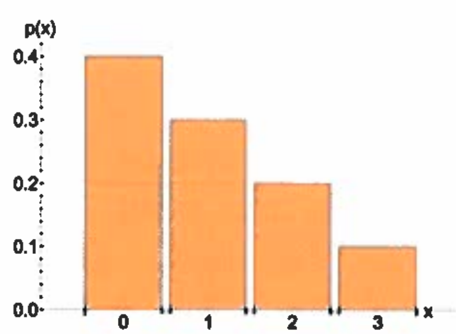
Due Date: March 17, 2023

1 Cumulative Distribution Functions

1. Find the cumulative distribution function for the discrete random variable that has the probability distribution function
$$f(x) = \frac{x}{15},$$
for $x = 1, 2, 3, 4, 5$.

2. If X is a discrete random variable with the cumulative distribution function
$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1/3 & \text{for } 1 \leq x < 4 \\ 1/2 & \text{for } 4 \leq x < 6 \\ 5/6 & \text{for } 6 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}.$$
find
   (a) $P(2 < X \leq 6)$;
   (b) $P(X = 4)$;
   (c) the probability distribution of X.

2 3. The probability distribution of X, the weekly number of accidents at a certain intersection, is plotted below.



   (a) Find the mean μ and the standard deviation σ for this distribution.
   (b) Construct the cumulative distribution of X and draw its graph.

4. A coin is biased so that heads is twice as likely as tails. For three independent tosses of the coin, let X denote the total number of heads.

   (a) Find the probability distribution of X and plot its graph.

   (b) Find the cumulative distribution of X and plot its graph.

   (c) Find $P(1 < X \leq 3)$ and $P(X > 2)$.

2 Continuous Random Variables

1. The probability density of the continuous random variable X is given by

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 2 < x < 7 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) Draw its graph and verify that the total area under the curve is equal to 1.

   (b) Find the probability distribution function of the random variable X.

   (c) Find $P(3 < X < 5)$.

2. (a) Show that

$$f(x) = 3x^2 \text{ for } 0 < x < 1$$

   represents a probability density function.

   (b) Sketch a graph of this function and indicate the area associated with $0.1 < x < 0.5$.

   (c) Calculate the probability that $0.1 < x < .5$.

3. (a) Show that

$$f(x) = e^{-x} \text{ for } 0 < x < \infty$$

   represents a probability density function.

   (b) Sketch a graph of this function and indicate the area associated with the probability that $x > 1$.

   (c) Calculate the probability that $x > 1$.

2 4. The probability density function of the random variable X is given by

$$f(x) = \begin{cases} \dfrac{c}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}.$$

   Find

   (a) the value of c;

   (b) $P(X < 1/4)$ and $P(X > 1)$;

   (c) the cumulative distribution of this random variable.

5. The probability density of the random variable Z is given by

$$f(z) = \begin{cases} kze^{-z^2} & \text{for } z > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) Find k and draw the graph of this probability density.

   (b) Find the cumulative distribution function of Z and draw its graph.

2 6. The total lifetime (in years) of five-year-old dogs of a certain breed is a random variable whose cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x \leq 5 \\ 1 - \dfrac{25}{x^2} & \text{for } x > 5 \end{cases}.$$

(a) Find the probabilty that such a five-year-old dog will live beyond 5 years.

(b) Find the probability that such a five-year-old dog will live less than eight years.

(c) Find the probability that such a five-year-old dog will live anywhere from 12 to 15 years.

(d) Find the probability density of this random variable.

3 Expected Values for Continuous Random Variables

2 1. If Y is a continuous random variable with density function $f(y)$, prove that

$$\sigma^2 = \mathbb{V}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2.$$

2. If Y is a continuous random variable with density function $f(y)$, mean μ, and variance σ^2 and a and b are constants prove that

(a) $\mathbb{E}(aY + b) = a\mu + b$.

(b) $\mathbb{V}(aY + b) = a^2\sigma^2$.

3. The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density function

$$f(y) = \begin{cases} cy^2(1-y)^4, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of c that makes $f(y)$ a probability density function.

(b) Find $\mathbb{E}(Y)$.

2 4. Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} \dfrac{3}{64}y^2(4-y), & 0 \leq y \leq 4, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the expected value and variance of weekly CPU time.

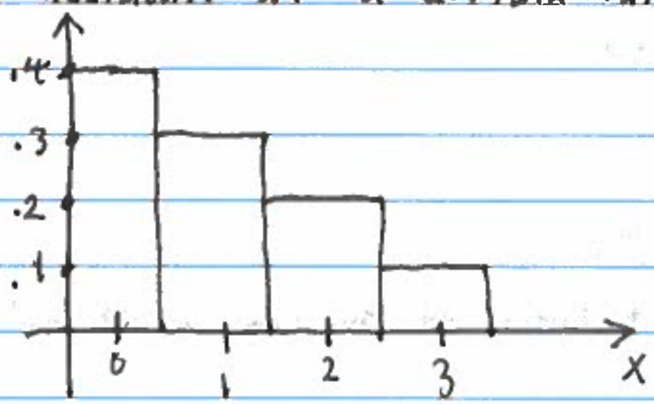
(b) The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

(c) Would you expect the weekly cost to exceed \$600 very often? Why?

Homework #7

#3.

The probability distribution of X, the weekly number of accidents at a certain intersection is plotted below



(a) Find the mean μ and the standard deviation σ for this distribution.

(b) Construct the cummulative distribution of X and draw its graph.

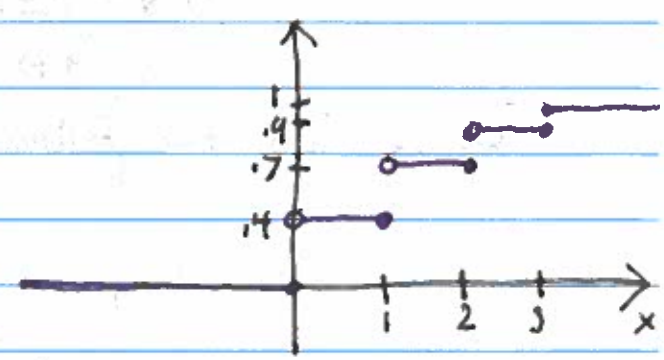
Solution:

(a) $\mathbb{E}[X] = \mu = .4 \cdot 0 + .3 \cdot 1 + .2 \cdot 2 + .1 \cdot 3 = .8$. Additionally, $\mathbb{E}[X^2] = .4 \cdot 0 + 1 \cdot .3 + 4 \cdot .2 + 9 \cdot .1 = 1.6$. Therefore,

$$\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1.1 - .64 = .96$$

$$\Rightarrow \sigma = .98.$$

(b). $$F(x) = \begin{cases} 0, & x \leq 0 \\ .4, & 0 < x \leq 1 \\ .7, & 1 < x \leq 2 \\ .9, & 2 < x \leq 3 \\ 1, & 3 < x \end{cases}$$



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#4.

The probability density function of the random variable X is given by

$$f(x) = \begin{cases} c x^{-1/2}, & 0 < x < 4 \\ 0, & \text{e.w.} \end{cases}$$

Find

(a) The value of c

(b) $P(X < 1/4)$ and $P(X > 1)$

(c) The cummulative distribution of this random variable

<u>Solution:</u>

(a) We know that $\int_{-\infty}^{\infty} f(x)dx = 1$ and thus $\int_0^4 c x^{-1/2} dx = 2c x^{1/2}\Big|_0^4 = 1$ implying $c = 1/4$.

(b) $P(X < 1/4) = \int_0^{1/4} \frac{1}{4} x^{-1/2} dx = \frac{1}{2} x^{1/2}\Big|_0^{1/4} = 1/4.$

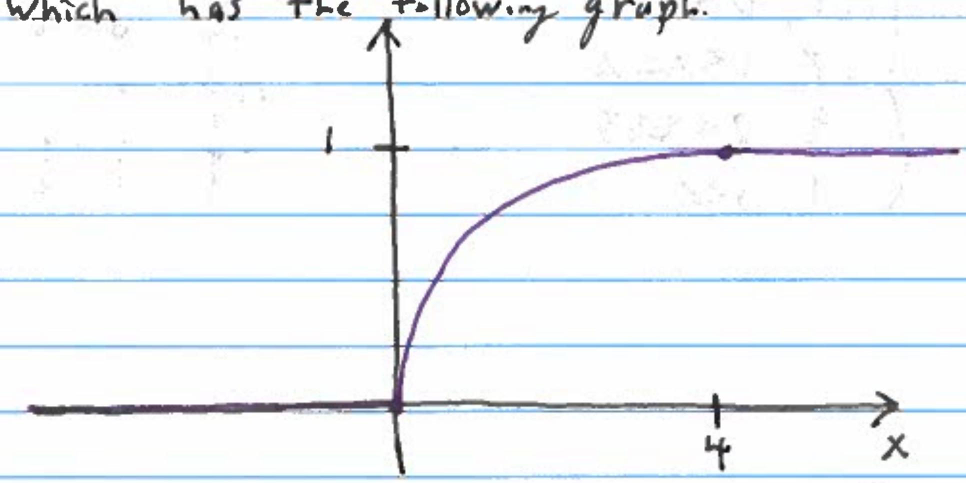
$P(X > 1) = \int_1^4 \frac{1}{4} x^{-1/2} dx = \frac{1}{2} x^{1/2}\Big|_1^4 = 1 - 1/2 = 1/2.$

(c) The cummulative distribution is given by

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$= \begin{cases} 0, & x \le 0 \\ 1/2 x^{1/2}, & 0 \le x \le 4 \\ 1, & 4 \le x \end{cases}$$

which has the following graph.



#6

The total lifetime of five-year-old dogs of a certain breed is a random variable whose cummulative distribution function is given by

$$F(x) = \begin{cases} 0, & x \le 5 \\ 1 - 25/x^2, & x > 5 \end{cases}$$

(a). Find the probability that such a five-year-old dog will live beyond 5 years.

(b) Find the probability that such a five-year-old will live less than eight years.

(c) Find the probability that such a five-year-old dog will live anywhere from 12 to 15 years.

(d) Find the probability density of this random variable.

Solution:

(a) $P(X > 5) = F(\infty) - F(5) = F(\infty) - 0 = 1.$

(b) $P(X < 8) = F(8) = 1 - 25/64 = 39/64.$

(c). $P(12 < X < 15) = F(15) - F(12) = 25/144 - 25/15^2 = 1/16.$

(d) $p(x) = \frac{dF}{dx}$

$$= \begin{cases} 0, & x \le 5. \\ 50/x^3, & x > 5 \end{cases}$$

#1

If Y is a continuous random variable with density function $f(y)$, prove that

$$\sigma^2 = \mathbb{V}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

proof:

$$\begin{aligned}
\mathbb{V}[Y] &= \mathbb{E}[(X-\mu)^2] \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \\
&= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\mu \int_{-\infty}^{\infty} x f(x)\,dx + \mu^2 \int_{-\infty}^{\infty} f(x)\,dx \\
&= \mathbb{E}[Y^2] - 2\mu\,\mathbb{E}[Y] + \mu^2 \\
&= \mathbb{E}[Y^2] - \mu^2 \\
&= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2
\end{aligned}$$

#4

Weekly CPU time used by an accounting firm has the probability density function (measured in hours) given by

$$f(y) = \begin{cases} \frac{3}{64} y^2 (4-y), & 0 \le y \le 4 \\ 0, & \text{e.w.} \end{cases}$$

(a) Find the expected time and variance of weekly CPU time.

(b) The CPU time costs the firm \$200.00 per hour. Find the expected value and variance of the weekly cost for CPU time.

(c) Would you expect the weekly cost to exceed \$600.00 very often? Why?

Solution:

(a) Calculating we have that

$$\mathbb{E}[Y] = \int_0^4 \tfrac{3}{64} y^3 (4-y)\,dy = \frac{3}{64} \int (4y^3 - y^4) = \frac{3}{64}\left(y^4 - y^5/5\right)\Big|_0^4 = 12/5 = 2.4$$

$$\mathbb{E}[Y^2] = \int_0^4 \tfrac{3}{64} y^4 (4-y)\,dy = \tfrac{3}{64} \int_0^4 (4y^4 - y^5)\,dy = \frac{3}{64}\left(\frac{4}{5} y^5 - y^6/6\right)\Big|_0^4 = 32/5$$

Consequently,

$$\mu = 12/5, \quad \sigma^2 = 32/5 - 144/25 = 16/25.$$

(b). Let $X = 200 \cdot Y$ denote the cost. Therefore,

$$\mathbb{E}[X] = 200 \cdot \mathbb{E}[Y] = 200 \cdot 12/5 = 480$$

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[200^2 Y^2] - 200^2 \mathbb{E}[X]^2 = 200^2 \sigma^2 = 200^2 \cdot 16/25$$

Therefore, the standard deviation for the cost is given by

$$\sigma_c = 200 \cdot 4/5 = \$160.00.$$

(c). We need to compute $P(Y \geq 3)$ which is given by

$$P(Y \geq 3) = \int_3^4 \frac{3}{64}(4y^2 - y^3)\,dy = \frac{3}{64}\left(\frac{4}{3}y^3 - \frac{1}{4}y^4\right)\Big|_3^4$$

$$\Rightarrow P(Y \geq 3) = \frac{3}{64}\left(\frac{4}{3} \cdot 4^3 - \frac{1}{4} \cdot 4^4 - \frac{4}{3} \cdot 3^3 + \frac{1}{4} 3^4\right) = 67/256 \approx .26$$

Consequently, we expect to exceed \$600.00 about 25% of the time.

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