MTH 357/657
Homework #8

Due Date: March 31, 2023

1 Moment Generating Functions

2 1. Suppose a and b are constants and X is a continuous random variable with moment generating function $m_X(t)$ and let $Y = X + a$, $Z = bX$, and $U = (X + a)/b$ be random variables with moment generating functions $m_Y(t)$, $m_Z(t)$, and $m_U(t)$ respectively. Show that the following identities are true:

   (a) $m_Y(t) = e^{at} m_X(t)$,

   (b) $m_Z(t) = m_X(bt)$,

   (c) $m_U(t) = e^{a/bt} m_X(t/b)$.

2 Gamma Distributions

1. Find the probabilities that the value of a random variable will exceed 4 if it has a gamma distribution with

   (a) $\alpha = 2$ and $\beta = 3$;

   (b) $\alpha = 3$ and $\beta = 4$.

2. Show that a gamma distribution with $\alpha > 1$ has a relative maximum at $x = \beta(\alpha - 1)$. What happens when $0 < \alpha < 1$ and when $\alpha = 1$?

3. Show that for $t < 1/\beta$ the moment generating function of the gamma distribution is given by

$$m(t) = (1 - \beta t)^{-\alpha}.$$

   Hint: Make the substitution $u = x(1/\beta - t)$ in the integral defining $m(t)$.

2 4. A random variable X has a Weibull distribution if and only if its probability density is given by

$$p(x) = \begin{cases} kx^{\beta-1}e^{-\alpha x^{\beta}} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases},$$

   where $\alpha > 0$ and $\beta > 0$.

   (a) Express k in terms of α and β.

   (b) Show that

$$\mu = \mathbb{E}[X] = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right).$$

2 5. If X is a random variable with an exponential distribution, i.e., a gamma distribution with $\alpha = 1$, and $P(X > 2) = .08$, determine the value of β and then compute $P(X \leq 1.5)$.

6. Suppose X is an exponential distribution with $\mu = \mathbb{E}[X] = 10$. Find the mean and variance of the following random variable:

$$C = 100 + 40X + 3X^2.$$

3 Beta Distributions

1. The percentage of impurities per batch in a chemical product is a random variable X with probability density function

$$p(x) = \begin{cases} kx^2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

A batch with more than 40% impurities cannot be sold.

(a) Find the value of k that makes this a probability density.

(b) Determine the probability that a randomly selected batch cannot be sold because of excessive impurities.

(c) Find the mean and variance of the of the percentage of impurities in a randomly selected batch of the chemical.

2 2. Prove that the variance of a beta-distributed random variable with parameters α and β is

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

2 3. For $\alpha, \beta > 1$, find the local maximum for the probability density function for the beta distribution in terms of α and β.

4 Normal Distribution

1. Show that a normal distribution has

(a) a relative maximum at $x = \mu$;

(b) inflection points at $x = \mu - \sigma$ and $x = \mu + \sigma$.

2 2. Assume that X is normally distributed with mean μ and standard deviation σ. After observing a value of X, a mathematician constructs a rectangle with length $L = |Y|$ and width $W = 3|Y|$. If A denotes the area of the resulting rectangle, calculate $\mathbb{E}(A)$.

Homework #8

#4

A random variable has a Weibull distribution if and only if its probability density is given by

$$p(x) = \begin{cases} kx^{\beta-1}e^{-\alpha x^{\beta}}, & x > 0 \\ 0, & x < 0 \end{cases}$$

Where $\alpha > 0$, $\beta > 0$.

(a) Express k in terms of α and β.

(b) Show that

$$\mu = \mathbb{E}[X] = \alpha^{-1/\beta}\Gamma(1 + 1/\beta).$$

Solution:

(a) $\int_{-\infty}^{\infty} p(x)dx = \int_0^{\infty} kx^{\beta-1}e^{-\alpha x^{\beta}}dx = 1.$

If we let $u = \alpha x^{\beta}$ we have that

$$du = \alpha\beta x^{\beta-1}dx$$

$$\Rightarrow \int_0^{\infty} kx^{\beta-1}e^{-\alpha x^{\beta}}dx = k\int_0^{\infty}\frac{1}{\alpha\beta}e^{-x}dx = k/\alpha\beta.$$

Therefore, $k = \alpha\beta$.

(b) Computing, we have that

$$\mathbb{E}[X] = \alpha\beta\int_0^{\infty} x^{\beta}e^{-\alpha x^{\beta}}dx.$$

Letting $u = \alpha x^{\beta}$ we have that $du = \alpha\beta x^{\beta-1}dx$ and $x = \alpha^{-1/\beta}u^{-1/\beta}$. Therefore,

$$\mathbb{E}[X] = \int_0^{\infty} xe^{-u}du$$

$$= \alpha^{-1/\beta}\int_0^{\infty} u^{-1/\beta}e^{-u}du$$

$$= \alpha^{-1/\beta}\Gamma(1 + 1/\beta).$$

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#5.

If X is a random variable with an exponential distribution and $P(X>2)=.08$, determine the value of β and then compute $P(X\le 1.5)$.

Solution:

The probability density for this random variable is given by

$$p(x)=1/\beta\, e^{-x/\beta}.$$

Therefore,

$$P(X>2)=1/\beta \int_2^\infty e^{-x/\beta}dx$$
$$=e^{-2/\beta}$$

Consequently,

$$e^{-2/\beta}=2/25$$
$$\Rightarrow -2/\beta=\ln(2/25)$$
$$\Rightarrow \beta=\frac{-2}{\ln(2/25)}=\frac{2}{\ln(25/2)}.$$

Finally,

$$P(X\le 1.5)=\int_0^{3/2} 1/\beta\, e^{-x/\beta}dx$$
$$=-e^{-x/\beta}\Big|_0^{3/2}$$
$$=1-e^{-3/2\beta}$$
$$=1-e^{-3/4\cdot\ln(25/2)}$$
$$=1-e^{\ln((25/2)^{-3/4})}$$
$$=1-\left(\frac{25}{2}\right)^{-3/4}$$
$$=1-\left(\frac{2}{25}\right)^{3/4}$$

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#2 Prove that the varience of a beta-distributed random variable with parameters α and β is

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Solution:

$$\mathbb{E}[X^2] = \frac{1}{\beta(\alpha,\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}dx$$

$$= \frac{\beta(\alpha+2,\beta)}{\beta(\alpha,\beta)} \cdot \frac{1}{\beta(\alpha+2,\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}dx$$

$$= \frac{\beta(\alpha+2,\beta)}{\beta(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$$

From class we have that $\mu = \mathbb{E}[X] = \alpha/\alpha+\beta$ and thus

$$\sigma^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha(\alpha^2+\alpha+\beta+\alpha\beta) - \alpha^3 - \alpha^2\beta - \alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

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#3

For $\alpha, \beta > 1$, find the local maximum for the probability density function for the beta distribution in terms of α and β.

Solution:

For $p(x) = K x^{\alpha-1}(1-x)^{\beta-1}$ we have that

$$p'(x) = K[(\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} - (\beta-1)x^{\alpha-1}(1-x)^{\beta-2}]$$
$$= K x^{\alpha-2}(1-x)^{\beta-2}[(\alpha-1)(1-x) - (\beta-1)x]$$

Therefore, the critical point satisfies

$$(\alpha-1)(1-x) - (\beta-1)x = 0$$
$$\Rightarrow (\alpha+\beta-2)x = \alpha-1$$
$$\Rightarrow x = \frac{\alpha-1}{\alpha+\beta-2}$$

Since $p(0) = p(1) = 0$ and $p(\alpha-1/\alpha+\beta-2) > 0$ it follows from the extreme value theorem that $\alpha-1/\alpha+\beta-2$ is the location of the global maximum.

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#2.

Assume that X is normally distributed with mean μ and standard deviation σ. After observing a value of X a mathematician constructs a rectangle with length $L = |X|$ and width $W = 3|X|$. If A denotes the area of the rectangle, calculate $\mathbb{E}[A]$.

Solution:

$$\mathbb{E}[A] = 3\,\mathbb{E}[X^2] = 3(\mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[X]^2)$$
$$= 3(\sigma^2 + \mu^2)$$

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#1

Suppose a and b are constants and X is a continuous random variable with moment generating function $m_X(t)$ and let $Y = X+a$, $Z = bX$, and $U=(X+a)/b$ be random variables with moment generating functions $m_Y(t)$, $m_Z(t)$, and $m_U(t)$ respectively. Show that the following are true

(a) $m_Y(t) = e^{at} m_X(t)$

(b) $m_Z(t) = m_X(bt)$

(c) $m_U(t) = e^{a/b\,t} m_X(t/b)$.

Solution:

(a)
$$\begin{aligned} m_Y(t) &= \mathbb{E}[e^{tY}] \\ &= \mathbb{E}[e^{t(X+a)}] \\ &= \mathbb{E}[e^{tX}e^{ta}] \\ &= e^{ta}\mathbb{E}[e^{tX}] \\ &= e^{ta} m_X(t). \end{aligned}$$

(b).
$$\begin{aligned} m_Z(t) &= \mathbb{E}[e^{tZ}] \\ &= \mathbb{E}[e^{tbX}] \\ &= m_X(tb) \end{aligned}$$

(c)
$$\begin{aligned} m_U(t) &= \mathbb{E}[e^{tU}] \\ &= \mathbb{E}[e^{t(X+a)/b}] \\ &= \mathbb{E}[e^{t/b\,X}e^{ta/b}] \\ &= e^{ta/b}\mathbb{E}[e^{t/b\,X}] \\ &= e^{ta/b} m_X(t/b). \end{aligned}$$

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