MTH 357/657

Homework #9

Due Date: April 07, 2023

1 Discrete Multivariate Distributions

1. The joint probability distribution $p(x, y)$ of random random variables X and Y satisfies

$$p(0,0) = \frac{1}{12},\ p(1,0) = \frac{1}{6},\ p(2,0) = \frac{1}{24},$$
$$p(0,1) = \frac{1}{4},\ p(1,1) = \frac{1}{4},\ p(2,1) = \frac{1}{40},$$
$$p(0,2) = \frac{1}{8},\ p(1,2) = \frac{1}{20},$$
$$p(0,3) = \frac{1}{120}.$$

   (a) Find the following values:
      i. $P(X = 1, Y = 2)$
      ii. $P(X = 0, 1 \leq Y < 3)$
      iii. $P(X + Y \leq 1)$
      iv. $P(X > Y)$

   (b) If $F(x, y)$ denotes the joint cumulative distribution function for this probability distribution, find the following values:
      i. $F(1.2, .9)$
      ii. $F(-3, 1.5)$
      iii. $F(2, 0)$
      iv. $F(4, 2.7)$

2. Suppose the joint probability distribution function of X and Y is given by

$$f(x, y) = c(x^2 + y^2)$$

   for $x = -1, 0, 1, 3$ and $y = -1, 2, 3$ and zero otherwise.

   (a) Find the value of c.
   (b) Compute $P(X \leq 1, Y > 2)$
   (c) Compute $P(X = 0, Y \leq 2)$
   (d) Compute $P(X + Y > 2)$

3. Show that there is no value of k for which

$$f(x, y) = ky(2y - x),\ \text{for } x = 0, 1, 2, 3 \text{ and } y = 0, 1, 2$$

   can serve as the joint probability distribution of two random variables.

2 Continuous Multivariate Distributions

1. If a radioactive particle is randomly located in a square of unit length, with coordinates (X, Y) selected randomly. A reasonable model for the joint density probability function for its X and Y is

$$f(x,y) = \begin{cases} 1 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) What is $P(X > .5, Y > .5)$?

   (b) What is $P(X - Y > .5)$?

   (c) What is $P(XY < 1/2)$?

2. Determine k so that

$$p(x,y) = \begin{cases} kx(x-y) & 0 < x < 1,\ -x < y < x \\ 0 & \text{elsewhere} \end{cases}$$

can serve as a joint probability density.

3. If the joint probability density of X and Y is given by

$$p(x,y) = \begin{cases} 24xy & 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & \text{elsewhere} \end{cases},$$

find $P\left(X + Y < \frac{1}{2}\right)$.

4. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 2 & \text{for } x > 0,\ y > 0,\ x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find

   (a) $P(X \le \frac{1}{2}, Y \le \frac{1}{2})$;

   (b) $P(X + Y > \frac{2}{3})$;

   (c) $P(X > 2Y)$.

5. pg. 234, #5.11.

3 Marginal and Conditional Distributions

1. With reference to problem #1 in the "Discrete Multivariate Distributions" section of this assignment, find

   (a) The marginal distribution of X;

   (b) The marginal distribution of Y;

   (c) The conditional distribution of X given $Y = 1$;

   (d) The conditional distribution of Y given $X = 0$;

2. Check whether the discrete random variables X and Y are independent if their joint probability distribution $f(x,y)$ is given by

   (a) $f(x,y) = 1/4$ for $x = -1$ and $y = -1$, $x = -1$ and $y = 1$, $x = 1$ and $y = -1$, and $x = 1$ and $y = 1$;

   (b) $f(x,y) = 1/3$ for $x = 0$ and $y = 0$, $x = 0$ and $y = 1$, and $x = 1$ and $y = 1$.

3. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find

   (a) the marginal density of X;

   (b) the conditional density of Y given $X = 1/4$;

   (c) the marginal density of Y;

   (d) the conditional density of X given $Y = 1$.

4. If the independent random variables X and Y have the marginal densities

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$g(y) = \begin{cases} \frac{1}{3} & \text{for } 0 < y < 3 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the joint probability density of X and Y.

   (b) Find $P(X^2 + Y^2 > 1)$.

5. Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y} & \text{for } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the probability that the sum of X and Y will exceed $1/2$.

   (b) Find the marginal density of X.

   (c) Find the marginal density of Y.

   (d) Determine whether the two random variables are independent.

Homework #9

#2

Suppose the joint probability distribution function of X and Y is given by

$$f(x,y) = c(x^2+y^2)$$

for $x=-1,0,1,3$ and $y=-1,2,3$ and zero otherwise.

(a) Find the value of c

(b) Compute $P(X \le 1, Y > 2)$

(c) Compute $P(X = 0, Y \le 2)$

(d) Compute $P(X+Y > 2)$.

Solution:

(a) $$\sum_x \sum_y c(x^2+y^2) = c\left(\sum_x \sum_y x^2 + \sum_x \sum_y y^2\right)$$

$$= 3c\sum_x x^2 + 4c\sum_y y^2$$

$$= 3c(1+1+9)+4c(1+4+9)$$
$$= 33c + 56c$$
$$= 89c.$$

Therefore, $c = 1/89$.

(b) $$P(X \le 1, Y > 2) = 1/89\left(\sum_{x\le 1} \sum_{y>2} x^2 + \sum_{x\le 1} \sum_{y>2} y^2\right)$$

$$= 1/89\left(\sum_{x\le 1} x^2 + 3\sum_{y>2} y^2\right)$$

$$= 1/89(1+1+3\cdot 9)$$
$$= 29/89.$$

(c) $P(X=0, Y\leq 2) = \frac{1}{89}\left(\sum_{y\leq 2} y^2\right)$

$$= \frac{1}{89}(1+4)$$
$$= \frac{5}{89}$$

(d) $P(X+Y>2) = \frac{1}{89}\left(\sum_{x}\sum_{y>2-x} x^2+y^2\right)$

$$= \frac{1}{89}\left(\sum_{y>3}(1+y^2) + \sum_{y>2}(0+y^2) + \sum_{y>1}(1+y^2) + \sum_{y>-1}(9+y^2)\right)$$
$$= \frac{1}{89}\left(0 + (0+9) + (1+4) + (1+9) + (9+4) + (9+9)\right)$$
$$= \frac{1}{89}(5\cdot 9 + 2\cdot 1 + 2\cdot 4)$$
$$= \frac{1}{89}(45+2+8)$$
$$= \frac{55}{89}.$$

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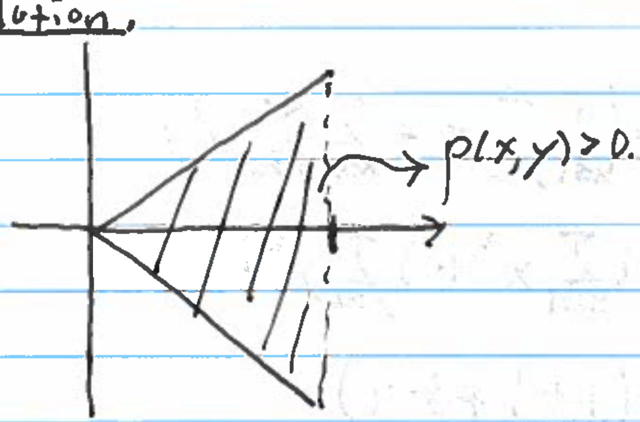
#2

Determine k so that

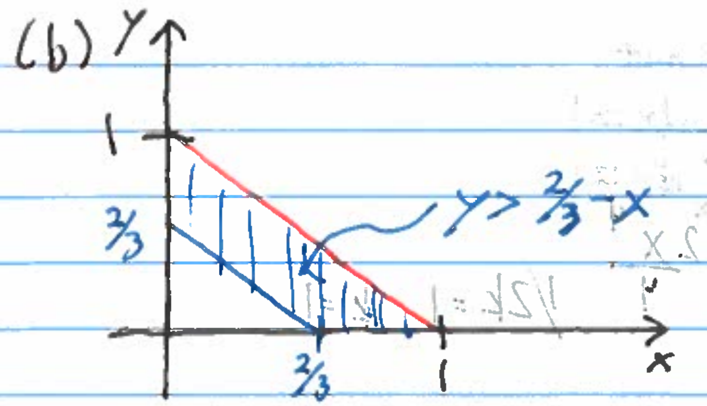
$$p(x,y) = \begin{cases} kx(x-y), & 0\leq x\leq 1, -x<y<x \\ 0, & \text{elsewhere} \end{cases}$$

can serve as a joint probability density.

Solution:



(b)



$$P(X+Y>2/3)=\int_0^{2/3}\int_{2/3-x}^{1-x} 2\,dy\,dx+\int_{2/3}^{1}\int_0^{1-x} 2\,dy\,dx$$

$$=2\int_0^{2/3}(1-x-2/3+x)\,dx+2\int_{2/3}^{1} 1-x\,dx$$

$$=2\int_0^{2/3} 1/3\,dx+2\int_{2/3}^{1}(1-x)\,dx$$

$$=2\cdot\frac{2}{9}+2(x-x^2/2)\Big|_{2/3}^{1}$$

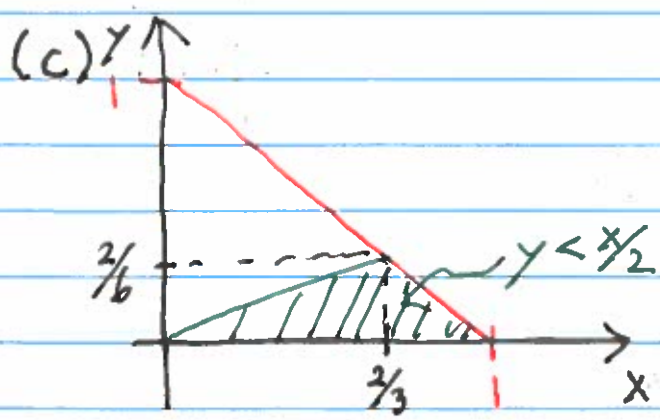
$$=4/9+2(1-1/2-2/3+4/18)$$

$$=4/9+2-1-4/3+4/9$$

$$=8/9-1/3$$

$$=5/9$$

(c)



The area of the triangle is 1/6 and thus

$$P(X>2Y)=\iint_A 2\,dy\,dx=2\iint_A dy\,dx=1/3.$$

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Integrating we have that

$$\int_0^1 \int_{-x}^{x} kx(x-y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^1 k\left(x^2 y - \frac{xy^2}{2}\right)\Big|_{-x}^{x} dx = 1$$

$$\Rightarrow \int_0^1 2kx^3\,dx = 1 \qquad \frac{2x^4}{4}$$

$$\Rightarrow \frac{2}{3}k = 1 \qquad \frac{1}{2}k = 1 \quad k = 1$$

$$\Rightarrow k = \frac{3}{2}. \quad k = 2\,?$$

∎

#4.

If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 2 & x>0,\ y>0,\ x+y<1 \\ 0 & \text{elsewhere} \end{cases}$$

find

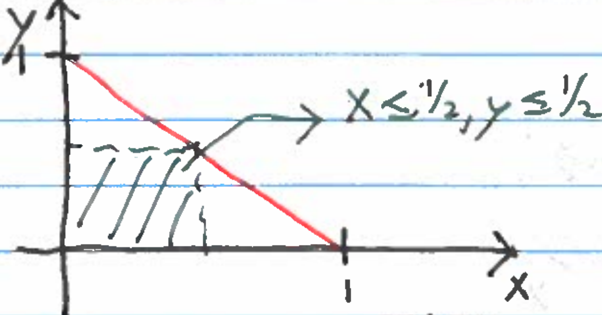
(a) $P(X \le \frac{1}{2}, Y \le \frac{1}{2})$

(b) $P(X + Y > \frac{2}{3})$

(c) $P(X > 2Y)$

<u>Solution:</u>

(a)



$$P(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}) = \int_0^{1/2}\int_0^{1/2} 2\,dy\,dx = 2 \cdot \tfrac{1}{4} = \tfrac{1}{2}.$$

#3.

If the joint probability density of X and Y is given by

$$p(x,y)=\begin{cases} \frac{1}{4}(2x+y), & 0<x<1,\ 0<y<2 \\ 0 & \text{, elsewhere} \end{cases}$$

find

(a) The marginal density of X;

(b) The conditional density of Y given $X=\frac{1}{4}$;

(c) The marginal density of Y;

(d) The conditional density of X given $Y=1$.

Solution:

(a)
$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} p(x,y)\,dy \\ &= \int_0^2 \tfrac{1}{4}(2x+y)\,dy, && \text{if } 0<x<1 \\ &= \tfrac{1}{4}\left(2xy+\tfrac{1}{2}y^2\right)\Big|_0^2, && \text{if } 0<x<1 \\ &= \tfrac{1}{4}(4x+2), && \text{if } 0<x<1 \end{aligned}$$

$$\Rightarrow f(x)=\begin{cases} \frac{1}{4}(4x+2), & 0<x<1 \\ 0, & \text{o.w.} \end{cases}$$

(b)
$$f(y|x=1)=\frac{p(x,\frac{1}{4})}{f(\frac{1}{4})}=\frac{p(x,\frac{1}{4})}{\frac{3}{4}}$$

$$\Rightarrow f(y|x=\tfrac{1}{4})=\begin{cases} \dfrac{\frac{1}{4}(\frac{1}{2}+y)}{\frac{3}{4}}, & 0<y<2 \\ 0 & \text{, elsewhere} \end{cases}$$

$$=\begin{cases} \frac{1}{6}+\frac{y}{3}, & 0<y<2 \\ 0 & \text{, elsewhere.} \end{cases}$$

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#4.

If the independent random variables X, Y have marginal densities

$$f(x)=\begin{cases} 1/2 & 0 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$g(y)=\begin{cases} 1/3 & 0 < y < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the joint probability density of X and Y.

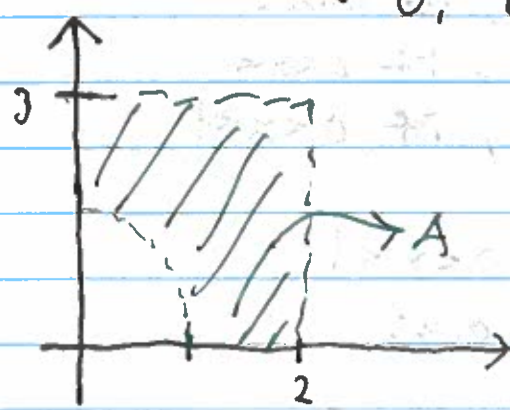
(b) Find $P(X^2+Y^2>1)$.

Solution:

(a) Since X, Y are independent it follows that

$$p(x,y)=f(x)g(y) = \begin{cases} 1/6, & 0<x<2,\ 0<y<3 \\ 0, & \text{elsewhere.} \end{cases}$$

(b)



$$P(X^2+Y^2>1) = \iint_A p(x,y)\,dy\,dx = \frac{1}{6}\iint_A dy\,dx = \left(6-\frac{\pi}{4}\right)/6 = 1-\pi/\cancel{6}\,24$$

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