MTH 357/657
Homework #6

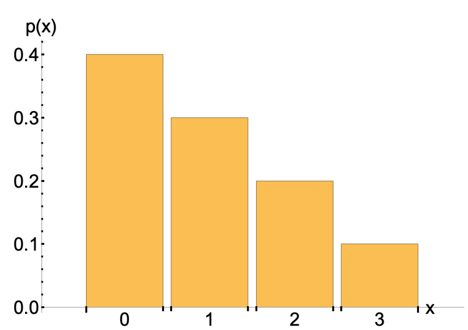
Due Date: March 17, 2023

1 Cumulative Distribution Functions

1. Find the cumulative distribution function for the discrete random variable that has the probability distribution function
$$f(x) = \frac{x}{15},$$
for $x = 1, 2, 3, 4, 5$.

2. If X is a discrete random variable with the cumulative distribution function
$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1/3 & \text{for } 1 \le x < 4 \\ 1/2 & \text{for } 4 \le x < 6 \\ 5/6 & \text{for } 6 \le x < 10 \\ 1 & \text{for } x \ge 10 \end{cases}.$$
find
   (a) $P(2 < X \le 6)$;
   (b) $P(X = 4)$;
   (c) the probability distribution of X.

3. The probability distribution of X, the weekly number of accidents at a certain intersection, is plotted below.



   (a) Find the mean μ and the standard deviation σ for this distribution.
   (b) Construct the cumulative distribution of X and draw its graph.

4. A coin is biased so that heads is twice as likely as tails. For three independent tosses of the coin, let X denote the total number of heads.
   (a) Find the probability distribution of X and plot its graph.
   (b) Find the cumulative distribution of X and plot its graph.
   (c) Find $P(1 < X \leq 3)$ and $P(X > 2)$.

2 Continuous Random Variables

1. The probability density of the continuous random variable X is given by
$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 2 < x < 7 \\ 0 & \text{elsewhere} \end{cases}.$$
   (a) Draw its graph and verify that the total area under the curve is equal to 1.
   (b) Find the probability distribution function of the random variable X.
   (c) Find $P(3 < X < 5)$.

2. (a) Show that
$$f(x) = 3x^2 \text{ for } 0 < x < 1$$
   represents a probability density function.
   (b) Sketch a graph of this function and indicate the area associated with $0.1 < x < 0.5$.
   (c) Calculate the probability that $0.1 < x < .5$.

3. (a) Show that
$$f(x) = e^{-x} \text{ for } 0 < x < \infty$$
   represents a probability density function.
   (b) Sketch a graph of this function and indicate the area associated with the probability that $x > 1$.
   (c) Calculate the probability that $x > 1$.

4. The probability density function of the random variable X is given by
$$f(x) = \begin{cases} \dfrac{c}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}.$$
   Find
   (a) the value of c;
   (b) $P(X < 1/4)$ and $P(X > 1)$;
   (c) the cumulative distribution of this random variable.

5. The probability density of the random variable Z is given by
$$f(z) = \begin{cases} kze^{-z^2} & \text{for } z > 0 \\ 0 & \text{elsewhere} \end{cases}.$$
   (a) Find k and draw the graph of this probability density.
   (b) Find the cumulative distribution function of Z and draw its graph.

6. The total lifetime (in years) of five-year-old dogs of a certain breed is a random variable whose cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x \leq 5 \\ 1 - \dfrac{25}{x^2} & \text{for } x > 5 \end{cases}.$$

   (a) Find the probabilty that such a five-year-old dog will live beyond 5 years.
   (b) Find the probability that such a five-year-old dog will live less than eight years.
   (c) Find the probability that such a five-year-old dog will live anywhere from 12 to 15 years.
   (d) Find the probability density of this random variable.

3 Expected Values for Continuous Random Variables

1. If Y is a continuous random variable with density function $f(y)$, prove that

$$\sigma^2 = \mathbb{V}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2.$$

2. If Y is a continuous random variable with density function $f(y)$, mean μ, and variance σ^2 and a and b are constants prove that
   (a) $\mathbb{E}(aY + b) = a\mu + b$.
   (b) $\mathbb{V}(aY + b) = a^2\sigma^2$.

3. The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density function

$$f(y) = \begin{cases} cy^2(1-y)^4, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Find the value of c that makes $f(y)$ a probability density function.
   (b) Find $\mathbb{E}(Y)$.

4. Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} \dfrac{3}{64} y^2(4-y), & 0 \leq y \leq 4, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Find the expected value and variance of weekly CPU time.
   (b) The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.
   (c) Would you expect the weekly cost to exceed \$600 very often? Why?