Lecture 10: Binomial, Geometric, Negative Binomial Distributions

Example:

What is the probability of rolling exactly 5 sixes in twelve rolls of a six sided die?

$$P(66666\overline{6666666}) = (1/6)^5(5/6)^7$$

the probability is the same for any rearangement.

$$\Rightarrow p(5) = \binom{12}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^7$$

Example:

Find the probability that exactly 3 out of 4 cars will pass inspection if the probability of passing an inspection is .80.

$$p(3) = P(PPPF) + P(PPFP) + P(PFPP) + P(FPPP)$$
$$= (.8)^3(.2) + (.8)^2(.2)(.8) + (.8)(.2)(.8)^2 + (.2)(.8)^3$$
$$= 4(.8)^3(.2)$$
$$= \binom{4}{3}(.8)^3(.2)$$

$\Rightarrow$ Compute prob. of one event times number of rearangements.

Definition - A random variable X is said to have a binomial distribution based on n trials with success probability p if and only if

$$p(x) = \binom{n}{x}p^x q^{n-x} = \binom{n}{x}p^x(1-p)^{n-x}$$

\# of rearrangements x-successes $n-x$ failures

$\mathbb{E}[X] = np \longrightarrow$ # of trials x prob. of success

Example:

In a box of 20 fuses, 6 are found to be defective. What is the most likely probability that a fuse is defective.

$\rightarrow$ Assume fuses are distributed with binomial distribution.

$$\Rightarrow P(X=6) = \binom{20}{6} p^6 (1-p)^{14} = f(p)$$

Find p that makes this number as large as possible.

$$\Rightarrow \frac{d}{dp} \ln(f(p)) = \frac{f'(p)}{f(p)} \Rightarrow f \text{ and } \ln(f) \text{ have same critical points } p^*$$

$$\Rightarrow \frac{d^2}{dp^2} \ln(f(p)) = \frac{f(p)f''(p) - f'(p)^2}{f(p)^2}$$

$$\Rightarrow \frac{d^2}{dp^2} \ln(f(p)) = \frac{f''(p^*)}{f(p^*)} \Rightarrow f'' \text{ and } \frac{d^2}{dp^2} \ln(f(p)) \text{ have same sign.}$$

$$\frac{d}{dp} \ln(f(p)) = \frac{d}{dp} \ln\left(\binom{20}{6} p^6 (1-p)^{14}\right)$$

$$= \frac{d}{dp} \ln\left[\binom{20}{6}\right] + \frac{d}{dp} \ln(p^6) + \frac{d}{dp} \ln(1-p)^{14}\Big]$$

$$= \frac{d}{dp} 6\ln(p) + \frac{d}{dp} 14\ln(1-p)$$

$$= \frac{6}{p} - \frac{14}{1-p}.$$

Critical point satisfies:

$$\frac{6}{p^*} = \frac{14}{1-p^*} \Rightarrow 6(1-p^*) = 14p^* \Rightarrow p^* = 3/10$$

$$\Rightarrow \frac{d^2}{dp^2} \ln(f(p)) = -\frac{6}{p^2} - \frac{14}{(1-p)^2} \Rightarrow p^* \text{ is a maximum.}$$

$\boxed{p^* = 3/10 \text{ is the most likely probability.}}$

Example:

If the probability is .40 that a child exposed to a disease will catch it, what is the probability that the tenth child exposed will be the third to catch it?

→ for the first nine children we need two to catch the disease

→ the tenth must catch it

$$\Rightarrow p = \underbrace{\binom{9}{2}(.4)^2(.6)^7}_{\text{probability two infected in first nine}} \underbrace{(.4)}_{\text{probability 10-th is infected.}} = \binom{9}{2}(.4)^3(.6)^7$$

Definition - A random variable X is said to have a negative binomial probability distribution if and only if

$$p(x) = \binom{x-1}{r-1} p^r q^{x-r}$$

* probability of r-th success on the x-th trial

$$\underbrace{\mu = \mathbb{E}[X] = \frac{r}{p}}_{\text{average number at which we expect the r-th success.}}$$

Example - If the probability is .25 that an applicant will pass their road test on the first try, how many times do we expect the candidate to take the exam?

$p(x) = (.75)^{x-1} p \rightarrow$ negative binomial with $r=1$

$\Rightarrow \mu = 4.$

Definition - A random variable X is said to have a geometric distribution if and only if

$$p(x) = q^{1-x} p, \quad \nu = 1/p.$$

Example: Suppose you poll students on whether they like mathematics or not and stop after you find the first student likes math. If the tenth person asked is the first to like math, what is the most likely percentage of students who like math.

Assume a geometric distribution

$$P(X=10) = (1-p)^9 p = f(p)$$

$$\frac{d}{dp} \ln(f(p)) = \frac{d}{dp} \ln((1-p)^9 p)$$

$$= \frac{d}{dp}(9\ln(1-p) + \ln(p))$$

$$= \frac{-9}{1-p} + \frac{1}{p}$$

$\Rightarrow p^*$ satisfies: $9p^* - 1 + p^* = 0 \Rightarrow p^* = \frac{1}{10}$