Lecture 14: Tchebysheff's Inequality

Theorem - If X is a non-negative random variable then

$$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a} \quad \text{(Markhov's Inequality)}$$

~~proof:~~

$$\mathbb{E}[X] = \sum_{x} x\,p(x) \geq \sum_{x \geq a} x\,p(x) \geq \sum_{x \geq a} a\,p(x) = a\,P(X \geq a)$$

$$\Rightarrow P(X \geq a) \leq \frac{1}{a}\mathbb{E}[X]$$

$\blacksquare$

Theorem - $P(|X-\mu| \geq k\sigma) \leq \dfrac{1}{k^2}$ (Chebychev's inequality)

$$\Rightarrow P(|X-\mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

~~proof~~

Apply Markhov's inequality to $(X-\mu)^2$:

$$P(|X-\mu| \geq k\sigma) = P(|X-\mu|^2 \geq k^2\sigma^2)$$
$$\leq \frac{\mathbb{E}(|X-\mu|^2)}{k^2\sigma^2}$$
$$= \frac{1}{k^2}.$$

$\blacksquare$

Why care?? This result gives an estimate of rare events and outliers.

$$\Rightarrow P(|X-\mu| \leq 3\sigma) = 1 - \tfrac{1}{9} = \tfrac{8}{9} \approx .89$$
$$P(|X-\mu| \leq 4\sigma) = 1 - \tfrac{1}{16} = \tfrac{15}{16} \approx .94$$

Example:

The number of customers per day at a sales counter is a random variable with $\mu = 20$ and $\sigma = 2$. What can be said about the probability that tomorrow between 16 and 24 customers will be served.

$$P(16 \le X \le 24) = P(|X - 20| \le 4)$$
$$= P(|X - 20| \le 2\sigma)$$
$$\ge 1 - \frac{1}{4}$$
$$= \frac{3}{4}.$$