# Lecture 17: Tchebysheff's Theorem

Theorem - Let $Y$ be a positive random variable. Then,

$$P(Y > a) \leq \frac{\mathbb{E}[Y]}{a}$$

where $a > 0$.

proof:

$$\begin{aligned}\mathbb{E}[Y] &= \int_0^\infty y\,p(y)\,dy \\ &\geq \int_a^\infty y\,p(y)\,dy \\ &\geq a\int_a^\infty p(y)\,dy \\ &= a\,P(Y > a)\end{aligned}$$

Therefore,

$$P(Y > a) \leq \frac{\mathbb{E}[Y]}{a}.$$

$\blacksquare$

Theorem - Let $X$ be a continuous random variable with $\mathbb{E}[X] = \mu$ and $\mathbb{E}[(X-\mu)^2] = \sigma^2$. Then

$$P(|X-\mu| > K\sigma) \leq 1/K^2$$

proof:

By Markhov's inequality

$$P(|X-\mu|^2 > K^2\sigma^2) \leq \frac{\mathbb{E}[|X-\mu|^2]}{K^2\sigma^2} = \frac{\sigma^2}{K^2\sigma^2} = \frac{1}{K^2}$$

$$\Rightarrow P(|X-\mu| > K\sigma) \leq \frac{1}{K^2}$$

Example:

Let $X$ be a random variable with probability density function

$$p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

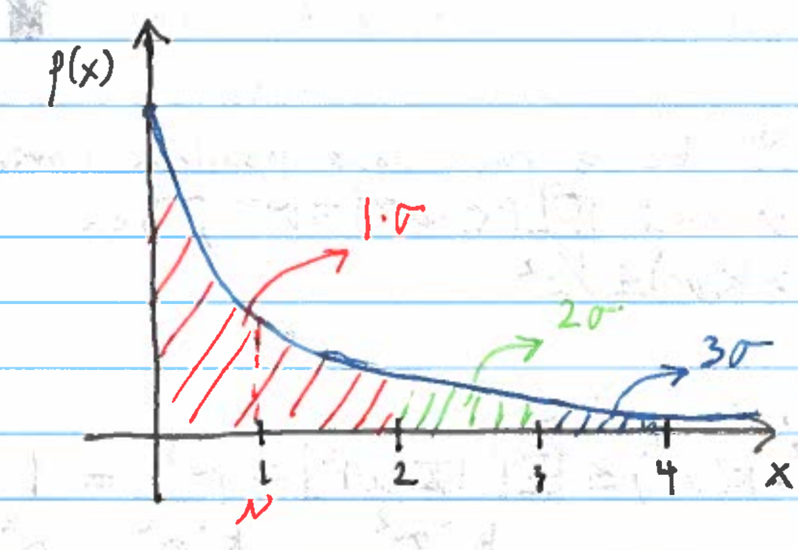
Therefore,

- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x\,p(x)\,dx = \int_0^{\infty} x e^{-x}\,dx = -xe^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\,dx = 1.$
- $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 p(x)\,dx = \int_0^{\infty} x^2 e^{-x}\,dx = 2$

$\Rightarrow \mu = 1, \quad \sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1$

By Tchebysheff's theorem

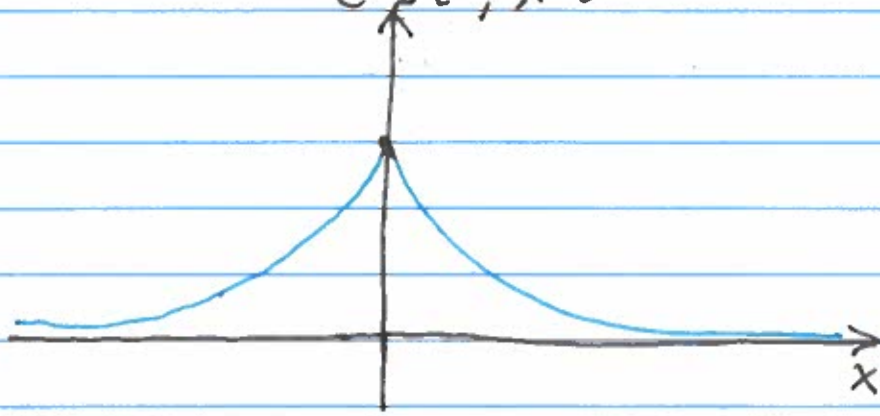
| | |
|---|---|
| $P(\lvert X-1\rvert \geq 1) < 1$ | $P(\lvert X-1\rvert > 1) = \int_1^{\infty} e^{-x}\,dx = e^{-1} = .37 < 1$ |
| $P(\lvert X-1\rvert \geq 2) < 1/4 = .25$ | $P(\lvert X-1\rvert \geq 2) = \int_2^{\infty} e^{-x}\,dx = e^{-2} = .14 < .25$ |
| $P(\lvert X-1\rvert \geq 3) < 1/9 \approx .11$ | $P(\lvert X-1\rvert \geq 3) = \int_3^{\infty} e^{-x}\,dx = e^{-3} = .05 < .11$ |



Example:

Let $X$ be a random variable with probability density function

$$p(x) = \frac{1}{2} e^{-|x|} = \begin{cases} \frac{1}{2} e^{x}, & x<0 \\ \frac{1}{2} e^{-x}, & x>0 \end{cases}$$



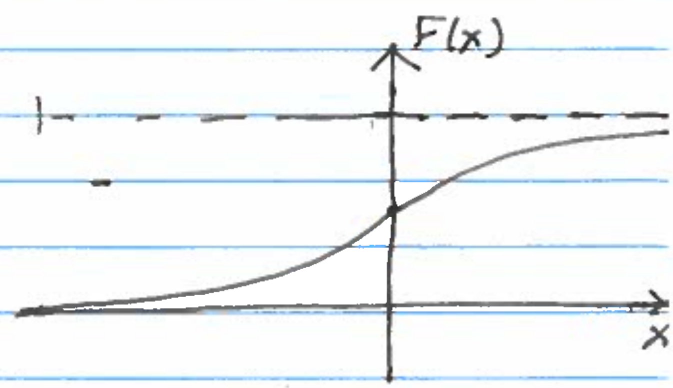

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, p(x)\, dx = 0$$ (area of odd function $=0$).

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 p(x)\, dx = 2\int_0^{\infty} x^2 \frac{1}{2} e^{-x} dx = 2$$

Therefore,

$$\mu = 0, \quad \sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2 \Rightarrow \mathbb{E}[X] = \sqrt{2}$$

$$F(x) = \int_{-\infty}^{x} p(s)\, ds$$
$$= \begin{cases} \frac{1}{2}\int_{-\infty}^{x} e^{s} ds, & x \le 0 \\ \frac{1}{2} + \frac{1}{2}\int_0^x e^{-s} ds, & x > 0 \end{cases}$$
$$= \begin{cases} \frac{1}{2} e^{x}, & x \le 0 \\ \frac{1}{2} - \frac{1}{2}(e^{-x} - 1), & x \ge 0 \end{cases}$$
$$= \begin{cases} \frac{1}{2} e^{x}, & x \le 0 \\ 1 - e^{-x}, & x \ge 0 \end{cases}$$



$P(|X| > \sqrt{2}) < 1$, $P(X > \sqrt{2}) = 2\int_{\sqrt{2}}^{\infty} \frac{1}{2} e^{-x} dx = e^{-\sqrt{2}} = .24$

$P(|X| > 2\sqrt{2}) < 1/4$ $P(X > 2\sqrt{2}) = \int_{2\sqrt{2}}^{\infty} e^{-x} dx = e^{-2\sqrt{2}} = .05$

$P(|X| \geq 3\sqrt{2}) \leq 1/9$ $P(X > 3\sqrt{2}) = \int_{3\sqrt{2}}^{\infty} e^{-x} dx = e^{-3\sqrt{2}} = .01$