Lecture 18: Moment Generating Functions and Γ-functions

Definition - The moment generating function of a continuous random variable X is given by
$$m(t) = \mathbb{E}[e^{tX}].$$

Theorem - If X is a continuous random variable then
$$\mu_k' = \mathbb{E}[X^k] = m^{(k)}(0).$$

~~proof~~:
If X is a continuous random variable with density $p(x)$ then
$$m(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} p(x)\,dx.$$
Therefore,
$$m^{(k)}(t) = \frac{d^k}{dt^k}\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} x^k e^{tx} p(x)\,dx$$
$$\Rightarrow m^{(k)}(0) = \int_{-\infty}^{\infty} x^k p(x)\,dx = \mathbb{E}[X^k].$$
$\blacksquare$

Example:
Find the moment generating function for the following probability density
$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases} \quad \text{(exponential density)}$$
where $\lambda > 0$.
$$\mathbb{E}[e^{tX}] = \int_0^{\infty} \lambda e^{tx} e^{-\lambda x}\,dx$$
$$= \int_0^{\infty} \lambda e^{(t-\lambda)x}\,dx$$
$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x}\Big|_0^{\infty}$$
$$= \frac{\lambda}{\lambda - t}.$$
$$= m(t).$$

Therefore,

$$m'(t) = \frac{\lambda}{(\lambda - t)^2}$$

$$m''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

$$\Rightarrow \mu_1' = 1/\lambda, \quad \mu_2' = 2/\lambda^2$$

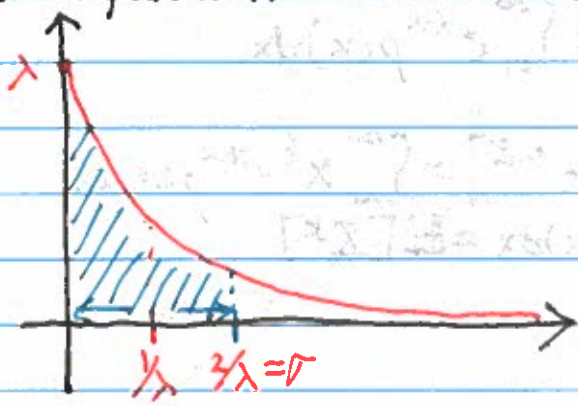
Consequently,

$$\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$$

and thus

$$\mu = 1/\lambda \text{ and } \sigma = 1/\lambda$$

for an exponential distribution.



$$P(|X - \mu| < \sigma) = \int_0^{2/\lambda} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{2/\lambda} = 1 - e^{-2} = .86$$

Definition - The gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\,dy.$$

Example:

1. $\Gamma(1) = \int_0^\infty y^0 e^{-y}\,dy = \int_0^\infty e^{-y}\,dy = 1$
2. $\Gamma(2) = \int_0^\infty y^1 e^{-y}\,dy = -ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y}\,dy = 1$
3. $\Gamma(3) = \int_0^\infty y^2 e^{-y}\,dy = -y^2 e^{-y}\Big|_0^\infty + \int_0^\infty 2y e^{-y}\,dy = 2$
4. $\Gamma(\alpha+1) = \int_0^\infty y^\alpha e^{-y}\,dy = -y^\alpha e^{-y}\Big|_0^\infty + \int_0^\infty \alpha y^{\alpha-1} e^{-y}\,dy$

   $\Rightarrow \Gamma(\alpha+1) = \alpha\Gamma(\alpha).$
5. $\Gamma(6) = 6\Gamma(5) = 6\cdot 5\cdot\Gamma(4) = 6! = 720$

Theorem - $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$

~~proof~~:

$$\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy = \left(\int_0^\infty e^{-x^2}\,dx\right)^2 = I^2$$

$x = r\cos\theta$, $y = r\sin\theta$ (convert to polar coordinates)

$$\Rightarrow \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \pi/2 \int_0^\infty r e^{-r^2}\,dr \quad , \quad u = r^2 \Rightarrow du = 2r\,dr$$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy = \pi/4 \int_0^\infty e^{-u}\,du = \pi/4.$$

Therefore,

$$\pi/4 = I^2$$

$$\Rightarrow I = \sqrt{\pi}/2.$$

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6. $\Gamma(1/2) = \int_0^\infty y^{-1/2} e^{-y}\, dy$

$y = u^2 \Rightarrow dy = 2u\,du = 2y^{1/2}\,du$

$\Rightarrow \Gamma(1/2) = \int_0^\infty 2e^{-u^2}\, du = \sqrt{\pi}.$

7. $\Gamma(3/2) = 1/2\, \sqrt{\pi}$

8. $\Gamma(5/2) = 3/2 \cdot 1/2\, \sqrt{\pi}$