Lecture 2: Probability of an Event

P1: The probability of an event $A \subset S$ is nonnegative:

$$P(A) \geq 0.$$

P2: $P(S) = 1$

P3: If $A_1, A_2, \ldots$ are mutually disjoint events then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \ldots$$

Example: If we flip a balanced coin twice, what is the probability of getting at least one head.

$$S = \{HH, HT, TH, TT\}$$

The event we get one head is

$$A = \{HH, HT, TH\}$$

$$P(A) = P(\{HH\} \cup \{HT\} \cup \{TH\})$$
$$= P(\{HH\}) + P(\{HT\}) + P(\{TH\})$$
$$= 1/4 + 1/4 + 1/4$$
$$= 3/4$$

Example: A die is loaded in such a way that each odd number is twice as likely as an even number. Find $P(G)$ where G is the event that a number greater than 3 is rolled.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let w denote the probability assigned to an even number. Therefore,

$$1 = P(S) = 2w + w + 2w + w + 2w + w$$
$$\Rightarrow 1 = 9w$$
$$\Rightarrow w = 1/9$$

Therefore,

$$P(G) = P(\{4\} \cup \{5\} \cup \{6\})$$
$$= P(\{4\}) + P(\{5\}) + P(\{6\})$$
$$= 1/9 + 2/9 + 1/9$$
$$= 4/9$$

Rules of Probability

1. $P(\bar{A}) = 1 - P(A)$

proof:

$$1 = P(S)$$
$$= P(A \cup \bar{A})$$
$$= P(A) + P(\bar{A})$$
$$\Rightarrow 1 - P(A) = P(\bar{A}).$$

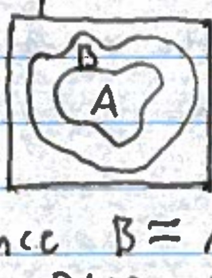
2. $P(\emptyset) = 0$

proof

$$P(S) = P(S \cup \emptyset)$$
$$= P(S) + P(\emptyset)$$
$$\Rightarrow 0 = P(\emptyset).$$

3. If $A \subseteq B$ then $P(A) \leq P(B)$.

proof:

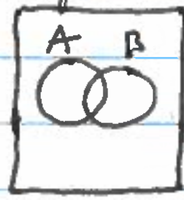


Since $B = A \cup (\bar{A} \cap B)$ we have that

$$P(B) = P(A \cup (\bar{A} \cap B))$$
$$= P(A) + P(\bar{A} \cap B)$$
$$\geq P(A).$$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

proof



$$P(A \cup B) = P((A \cap \bar{B}) \cup (A \cap B) \cup (B \cap \bar{A}))$$
$$= P(A \cap \bar{B}) + P(A \cap B) + P(B \cap \bar{A})$$
$$= P(A \cap \bar{B}) + P(A \cap B) + P(B \cap \bar{A}) + P(B \cap A) - P(B \cap A)$$
$$= P(A) + P(B) - P(B \cap A).$$

Examples

Exit on a highway

.23 = probability truck stopped has faulty brakes

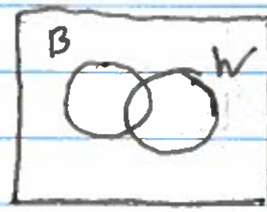
.24 = probability of worn tires

.38 = probability faulty brakes or worn tires

$\Rightarrow P(B) = .23$

$P(W) = .24$

$P(B \cup W) = .38$



What is the probability the tires are worn and the brakes are faulty.

$$.38 = .24 + .23 - P(B \cap W)$$
$$\Rightarrow P(B \cap W) = .47 - .38 = .09$$

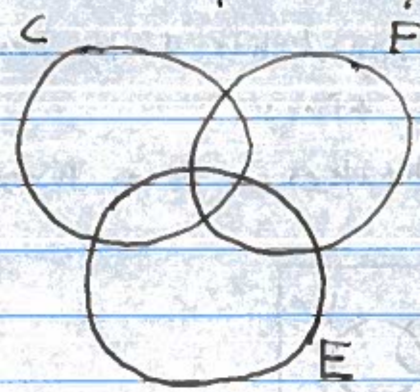
Example:

Dentist visit

.44 = probability teeth cleaned, C
.24 = probability cavity filled, F
.21 = probability tooth extracted, E

$P(C \cap F) = .08$
$P(C \cap E) = .11$
$P(F \cap E) = .07$
$P(C \cap F \cap E) = .03$

What is the probability at least one of these events occur?



$P(C \cup F \cup E) = P(C) \cup P(F) \cup P(E) - P(C \cap F) - P(C \cap E) - P(F \cap E) + P(C \cap F \cap E)$
$= .44 + .24 + .21 - .08 - .11 - .07 + .03$
$= .66$