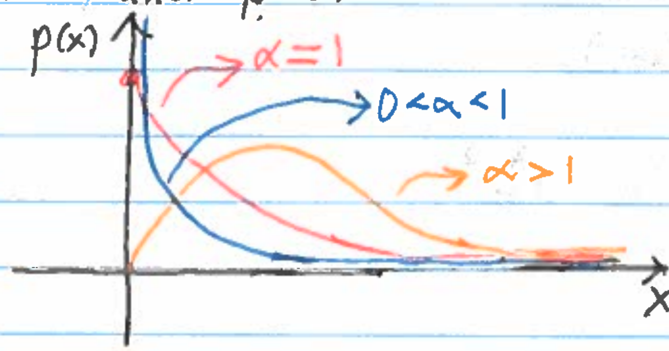
<u>Lecture 20: The Γ-distribution.</u>

Definition - A random variable has a gamm distribution and its probability density is given by

$$p(x) = \begin{cases} K x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{e.w.,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.



1. $\int_{-\infty}^{\infty} p(x)dx = \int_0^{\infty} K x^{\alpha-1} e^{-x/\beta} dx, \quad (u = x/\beta, du = dx/\beta)$

$$= \int_0^\infty K \beta^\alpha u^{\alpha-1} e^{-u} du$$

$$= K \beta^\alpha \Gamma(\alpha)$$

$$\Rightarrow K = \frac{1}{\beta^\alpha \Gamma(\alpha)}$$

2. $\mathbb{E}[X] = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^\alpha e^{-x/\beta} dx \quad (u = x/\beta, du = dx/\beta)$

$$= \frac{\beta^{\alpha+1}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty u^\alpha e^{-u} du$$

$$= \beta \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \beta\alpha.$$

3. $\mathbb{E}[X^2] = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-x/\beta} dx \quad (u = x/\beta,\ du = 1/\beta\, dx)$

$$= \frac{\beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty u^{\alpha+1} e^{-u} du$$

$$= \frac{\beta^2 \Gamma(\alpha+2)}{\Gamma(\alpha)}$$

$$= \beta^2 (\alpha+1)\alpha.$$

4. $\sigma^2 = \beta^2(\alpha+1)\alpha - \beta^2\alpha^2$

$$= \beta^2 \alpha$$

Definition - A random variable X has a chi-square distribution with ν degrees of freedom if and only if X is gamma distributed with $\alpha = \nu/2$ and $\beta = 2$

$$\Rightarrow \mu = \nu$$

$$\sigma^2 = 2\nu$$