Lecture 22: Multivariate Distributions

Example:

Two caplets are selected randomly from a bottle containing three aspirin, two sedatives, and four placebos. If X and Y are the number of aspirin and sedatives selected, find the probabilities associated with each pair of values of X and Y.

$$S = \{(0,0), (1,0), (0,1), (1,1), (2,0), (0,2)\}$$
$$|S| = 6$$

- $P(X=0, Y=0) = p(0,0) = \dfrac{\binom{3}{0}\binom{2}{0}\binom{4}{2}}{\binom{9}{2}} = \dfrac{6}{36} = \dfrac{1}{6}$
- $P(X=1, Y=0) = p(1,0) = \dfrac{\binom{3}{1}\binom{2}{0}\binom{4}{1}}{\binom{9}{2}} = \dfrac{12}{36} = \dfrac{1}{3}$
- $P(X=0, Y=1) = p(0,1) = \dfrac{\binom{3}{0}\binom{2}{1}\binom{4}{1}}{\binom{9}{2}} = \dfrac{8}{36} = \dfrac{2}{9}$
- $P(X=1, Y=1) = p(1,1) = \dfrac{\binom{3}{1}\binom{2}{1}\binom{4}{0}}{\binom{9}{2}} = \dfrac{6}{36} = \dfrac{1}{6}$
- $P(X=2, Y=0) = p(2,0) = \dfrac{\binom{3}{2}\binom{2}{0}\binom{4}{0}}{\binom{9}{2}} = \dfrac{3}{36} = \dfrac{1}{12}$
- $P(X=0, Y=2) = p(0,2) = \dfrac{\binom{3}{0}\binom{2}{2}\binom{4}{0}}{\binom{9}{2}} = \dfrac{1}{36}$

Joint probability distribution

$$\Rightarrow p(x,y) = P(X=x, Y=y) = \frac{\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}}{\binom{9}{2}},$$

for $x = 0,1,2$; $y = 0,1,2$)
$0 \le x+y \le 2$

y \ x	0	1	2
0	1/6	1/3	1/12
1	2/9	1/6	0
2	1/36	0	0

Theorem - A bivariate function can serve as the joint probability distribution for a pair of random variables X, Y if and only if its values, $f(x,y)$, satisfy the conditions

1. $f(x,y) \geq 0$
2. $\sum_x \sum_y f(x,y) = 1.$

Definition - The cumulative distribution function $F(x,y)$ is defined by

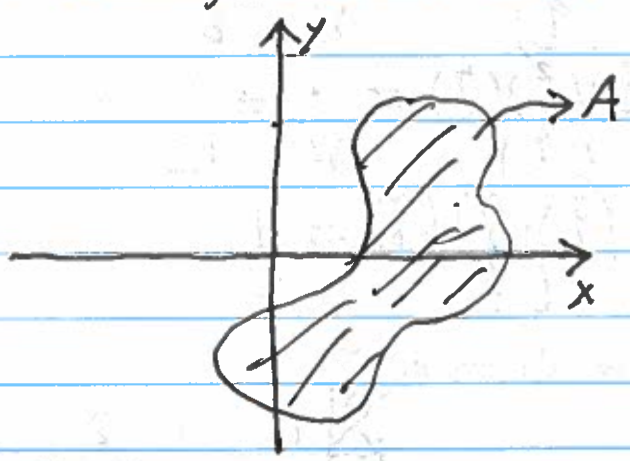
$$F(x,y) = P(X \leq x, Y \leq y)$$

Example: - Returning to the bottle example,

$$\begin{aligned} F(1,1) &= P(X \leq 1, Y \leq 1) \\ &= f(0,0) + f(0,1) + f(1,0) + f(1,1) \\ &= 1/6 + 2/9 + 1/3 + 1/6 \\ &= 8/9. \end{aligned}$$

Continuous Case:

Let X, Y be two continuous random variables.



$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$

joint probability density function.

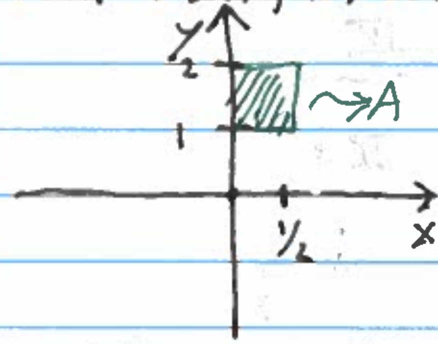
1. $f(x,y) \geq 0, \ -\infty < x < \infty, \ -\infty < y < \infty$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

Example:

Given the joint probability density function

$$f(x,y)=\begin{cases} \frac{3}{5}x(y+x), & 0<x<1,\ 0<y<2 \\ 0, & \text{e.w.} \end{cases}$$

find $P[(X,Y)\in A]$, where $A=\{(x,y): 0<x<\frac{1}{2}, 1<y<2\}$.



$$\Rightarrow P[(X,Y)\in A]=\int_1^2\int_0^{1/2} \frac{3}{5}x(y+x)\,dx\,dy$$
$$=\frac{3}{5}\int_1^2 \left.\frac{1}{2}x^2y+\frac{1}{3}x^3\right|_{x=0}^{x=1/2}$$
$$=\frac{3}{5}\int_1^2\left(\frac{1}{8}y+\frac{1}{24}\right)dy$$

$$\Rightarrow P[(X,Y)\in A]=\frac{3}{5}\left.\left(\frac{1}{16}y^2+\frac{1}{24}y\right)\right|_1^2$$
$$=\frac{3}{5}\left(\frac{1}{16}\cdot 4+\frac{1}{24}\cdot 2-\frac{1}{16}-\frac{1}{24}\right)$$
$$=\frac{3}{5}\left(\frac{3}{16}+\frac{1}{24}\right)$$
$$=\frac{3}{40}\left(\frac{3}{2}+\frac{1}{3}\right)$$
$$=\frac{3}{40}\left(\frac{11}{6}\right)$$
$$=\frac{11}{80}.$$

Definition- If X and Y are continuous random variables, the function given by

$$F(x,y)=P(X\le x, Y\le y)$$
$$=\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt$$

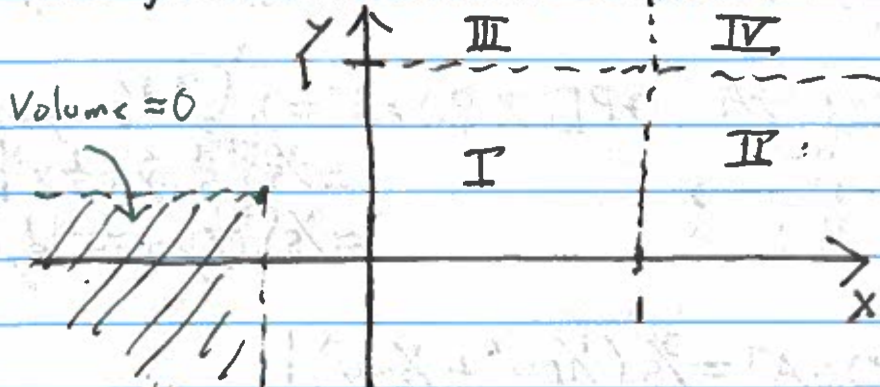
is called the joint distribution function of X and Y.

Example:

If the joint density of X and Y is given by

$$f(x,y)=\begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{e.w.} \end{cases}$$

find the joint distribution function.



$$F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt$$

1. If $x,y<0 \Rightarrow F(x,y)=0$
2. If $0<x<1$ and $0<y<1$

$$\begin{aligned} F(x,y) &= \int_0^y\int_0^x (s+t)\,ds\,dt \\ &= \int_0^y \left(s^2/2 + ts\Big|_0^x\right)dt \\ &= \int_0^y \left(x^2/2 + tx\right)dt \\ &= \left(x^2/2\,t + t^2/2\cdot x\right)\Big|_0^y \\ &= x^2/2\,y + y^2/2\cdot x \\ &= \frac{xy}{2}(x+y). \end{aligned}$$

3. If $x>1$ and $0<y<1$

$$\begin{aligned} F(x,y) &= \int_0^y\int_0^1 (s+t)\,ds\,dt \\ &= \int_0^y (s^2/2 + st)\Big|_0^1\,dt \\ &= \int_0^y (1/2 + t)\,dt \\ &= (1/2\,t + t^2/2)\Big|_0^y \\ &= 1/2\,y(y+1) \end{aligned}$$

4. If $0<x<1$ and $y>1$

$$F(x,y)=\frac{1}{2}x(1+x)$$

5. If $x>1,\ y>1$ we have

$$F(x,y)=\int_0^1\int_0^1(s+t)\,dt\,ds=1.$$

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Example:

Find the joint density of two random variables X and Y whose joint distribution is given by

$$F(x,y)=\begin{cases}(1-e^{-x})(1-e^{-y}) & \text{for } x>0,\ y>0\\ 0 & \text{elsewhere}\end{cases}$$

$$(1-e^{-x})(1-e^{-y})=\int_0^y\int_0^x p(x,y)\,dx\,dy$$

$$\Rightarrow \frac{d}{dx}\frac{d}{dy}F(x,y)=p(x,y)$$

$$\Rightarrow \frac{d}{dx}(1-e^{-x})e^{-y}=p(x,y)$$

$$\Rightarrow e^{-x}e^{-y}=p(x,y)$$

Therefore,

$$p(x,y)=\begin{cases}e^{-(x+y)}, & x>0 \text{ and } y>0\\ 0, & \text{elsewhere}\end{cases}$$

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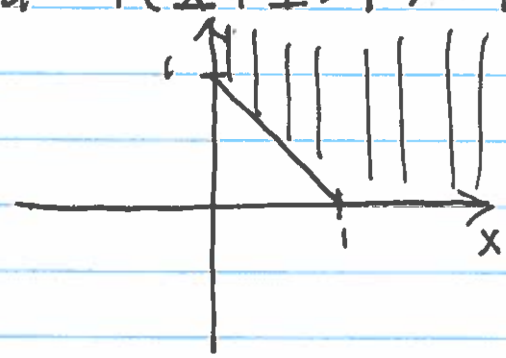
Find $P(X+Y>1)=P(Y>1-X)=P(X>1-Y)$



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\begin{aligned}
P(X+Y>1) &= \int_0^1 \int_{1-t}^{\infty} e^{-(s+t)}\,ds\,dt + \int_1^{\infty}\int_0^{\infty} e^{-(s+t)}\,ds\,dt \\
&= \int_0^1 e^{-t}(-e^{-s})\Big|_1^{\infty}\,dt + \int_1^{\infty} e^{-t}(-e^{-s})\Big|_1^{\infty} \\
&= \int_0^1 e^{-t}e^{-1}\,dt + \int_1^{\infty} e^{-t}\,dt \\
&= -e^{-t}e^{-1}\Big|_0^1 - e^{-t}\Big|_1^{\infty} \\
&= -e^{-2}+1-e^{-1}.
\end{aligned}
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