Lecture 23: Marginal and Conditional Distributions

Example:

X, Y are discrete random variables with the following joint distribution

$y \backslash x$	0	1	2	Σ
0	$1/6$	$1/3$	$1/12$	$7/12 = P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0) + P(X=2, Y=0)$
1	$2/9$	$1/6$	0	$7/18 = P(Y=1)$
2	$1/36$	0	0	$1/36 = P(Y=2)$
Σ	$\frac{5}{12}$	$1/2$	$1/2$	

Since $\sum_x \sum_y p(x,y) = \sum_y \sum_x p(x,y) = 1$ it follows that

$$g(x) = \sum_y p(x,y), \qquad h(y) = \sum_x p(x,y),$$

definition of marginal distributions.

are probability distributions called marginal distributions

Definition - If X, Y are continuous random variables with joint probability density $p(x,y)$ the functions

$$g(x) = \int_{-\infty}^{\infty} p(x,y)\,dy$$
$$h(y) = \int_{-\infty}^{\infty} p(x,y)\,dx$$

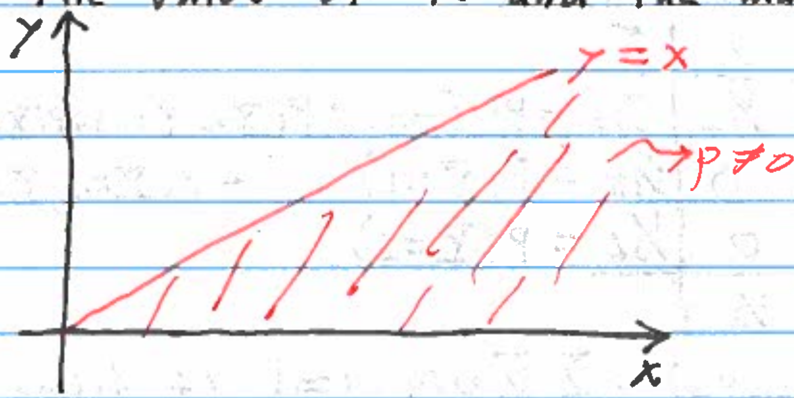
are called the marginal densities of X and Y respectively.

Example:

Suppose

$$p(x,y) = \begin{cases} ke^{-(x+y)}, & 0<x<y \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of k and the marginal distributions.



1. $\int_0^\infty \int_0^x ke^{-(x+y)}\,dy\,dx = \int_0^\infty -ke^{-(x+y)}\Big|_{y=0}^{y=x} dx$

$$= -\int_0^\infty (ke^{-2x} - ke^{-x})\,dx$$

$$= k\int_0^\infty (e^{-x} - e^{-2x})\,dx$$

$$= k(-e^{-x} + \tfrac{1}{2}e^{-2x})\Big|_0^\infty$$

$$= k(1 - \tfrac{1}{2})$$

$$= 1$$

$\Rightarrow k = 2$

2. $g(x) = \int_{-\infty}^{\infty} 2e^{-(x+y)}\,dy = \int_0^x 2e^{-(x+y)}\,dy = 2e^{-x}\int_0^x e^{-y}\,dy = -2e^{-x}e^{-y}\Big|_0^x$

$$\Rightarrow \boxed{g(x) = \begin{cases} 2(e^{-x} - e^{-2x}), & x>0 \\ 0, & x\le 0 \end{cases}}$$

3. $h(y) = \begin{cases} \int_y^\infty 2e^{-(x+y)}\,dx, & y>0 \\ 0, & y<0 \end{cases}$

$$= 2e^{-y}\begin{cases} -e^{-x}\Big|_y^\infty, & y>0 \\ 0, & y<0 \end{cases}$$

$$\Rightarrow h(y) = 2e^{-y}\begin{cases} e^{-y}, & y>0 \\ 0, & y\le 0 \end{cases}$$

$$\Rightarrow \boxed{h(y) = \begin{cases} 2e^{-2y}, & y>0 \\ 0, & y\le 0 \end{cases}}$$

Return to Conditional Probabilities

Recall:

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(A|B)\,P(B).$$

For discrete random variables:

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$= \frac{p(x,y)}{h(y)}$$

$\rightarrow$ marginal distribution of y.

Definition - If X, Y are discrete random variables with joint distribution $p(x,y)$ then

$$f(x|y) = \frac{p(x,y)}{h(y)}, \quad w(y|x) = \frac{p(x,y)}{g(x)}$$

are called conditional distributions.

Definition - If X, Y are continuous random variables with joint density $p(x,y)$ then

$$f(x|y) = \frac{p(x,y)}{h(y)}, \quad w(y|x) = \frac{p(x,y)}{g(x)}$$

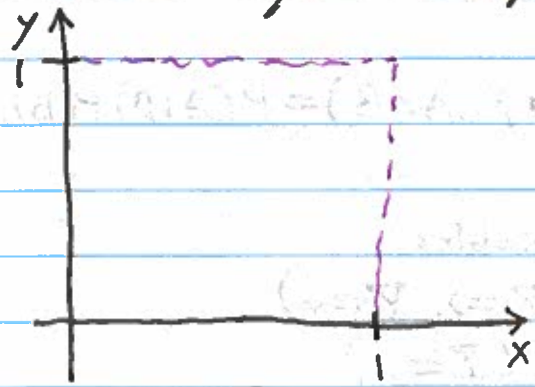
are the conditional density functions.

Example

Given the joint density

$$f(x,y) = \begin{cases} 4xy, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

find the marginal densities of X and Y and the conditional densities $f(x|y)$ and $f(y|x)$.



$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \begin{cases} \int_0^1 4xy\,dy, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$
$$= \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \begin{cases} \int_0^1 4xy\,dx, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$
$$= \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{4xy}{2y} = 2x = g(x) \qquad (0 \le x \le 1,\ 0 \le y \le 1)$$

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{4xy}{2x} = 2y = h(y) \qquad (0 \le x \le 1,\ 0 \le y \le 1)$$

Notice, in the previous example we have that

$$f(x,y) = f(x|y)h(y) = g(x)h(y)$$
$$f(x,y) = f(y|x)g(x) = h(y)g(x).$$

Definition - If $f(x_1, \dots, x_n)$ is the joint density of random variables $X_1, \dots, X_n$ then $X_1, \dots, X_n$ are independent if and only if

$$f(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n).$$

Example:

Given independent random variables X_1, X_2, X_3 with densities

$$f_1(x_1) = \begin{cases} e^{-x_1}, & x_1 > 0 \\ 0, & \text{e.w.} \end{cases}, \quad f_2(x_2) = \begin{cases} 2e^{-2x_2}, & x_2 > 0 \\ 0, & \text{e.w} \end{cases}$$

$$f_3(x_3) = \begin{cases} 3e^{-3x_3}, & x_3 > 0 \\ 0, & \text{e.w.} \end{cases}$$

find their joint probability distribution and

$P(X_1 + X_2 \leq 1, X_3 > 0)$

$$\Rightarrow f(x_1, x_2, x_3) = \begin{cases} 6e^{-(x_1 + 2x_2 + 3x_3)}, & x_1, x_2, x_3 > 0 \\ 0 & \text{e.w.} \end{cases}$$

$$\Rightarrow P(X_1 + X_2 \leq 1, X_3 > 0) = \int_0^\infty \int_0^1 \int_0^{1-x_1} 6e^{-x_1} e^{-2x_2} e^{-3x_3} \, dx_2 \, dx_1 \, dx_3$$
$$= \int_0^\infty \int_0^1 6e^{-x_1} e^{-3x_3} \left(-\tfrac{1}{2} e^{-2x_2}\right)\Big|_0^{1-x_1} dx_1 \, dx_3$$
$$= \int_0^\infty \int_0^1 3e^{-x_1} e^{-3x_3} \left(1 - e^{-2(1-x_1)}\right) dx_1 \, dx_3$$
$$= (1 - 2e^{-1} + e^{-2}) e^{-3}$$
$$= .02$$

Example:

If X, Y are continuous random variables then X, Y are independent if and only if

$$p(x,y) = f(x)g(y)$$

(f and g do not have to be probability densities)

~~proof~~:

$$p(x,y) = K f(x) \frac{1}{K} g(y)$$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K f(x) \frac{1}{K} g(y)\,dy$$

$$= \int_{-\infty}^{\infty} K f(x)\,dx \int_{-\infty}^{\infty} \frac{1}{K} g(y)\,dy$$

Set K so that

- $\int_{-\infty}^{\infty} \frac{1}{K} g(y)\,dy = 1 \Rightarrow \int_{-\infty}^{\infty} K f(x)\,dx = 1$
- $\int_{-\infty}^{\infty} K f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{1}{K} g(y)\,dy = 1.$

Example

- $p(x,y) = \begin{cases} 4xy, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$

$$= f(x)g(y)$$

$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{o.w.} \end{cases}, \quad g(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{o.w} \end{cases}$

$\Rightarrow X, Y$ are independent.

- $p(x,y) = \begin{cases} x^2 + y^2, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$

$\Rightarrow X, Y$ are dependent.