Lecture 25: Distribution Function Technique

Example:

If the probability density of X is given by
$$p(x) = \begin{cases} 6x(1-x), & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$$
find the probability density of $Y = X^3$.

Idea: $F(y) = P(Y \le y)$

$\Rightarrow g(y) = \frac{dF}{dy}$ is the probability density of Y.

$$\begin{aligned} \Rightarrow F(y) &= P(Y \le y) \\ &= P(X^3 \le y) \\ &= P(X \le y^{1/3}) \\ &= \int_{-\infty}^{y^{1/3}} p(x)\,dx \end{aligned}$$

$$= \begin{cases} 0, & y<0 \\ \int_0^{y^{1/3}} 6x(1-x)\,dx, & 0 \le y \le 1 \\ 1, & 1<y \end{cases}$$

$$= \begin{cases} 0, & y<0 \\ 3x^2 - 2x^3 \Big|_0^{y^{1/3}}, & 0 \le y \le 1 \\ 1, & 1<y \end{cases}$$

$$= \begin{cases} 0, & y<0 \\ 3y^{2/3} - 2y, & 0 \le y \le 1 \\ 1, & 1<y \end{cases}$$

$$\Rightarrow g(y) = \begin{cases} 2y^{-1/3} - 2, & 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Example:

If $Y = |X|$ show that the probability density for Y is given by

$$g(y) = \begin{cases} p(y) + p(-y), & y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

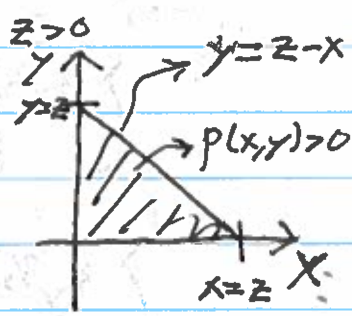
$$F(y) = P(Y \le y)$$
$$= P(|X| \le y)$$
$$= \begin{cases} P(-y < x < y), & y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$
$$= \begin{cases} \int_{-y}^{y} p(x)\,dx, & y > 0 \\ 0, & \text{elsewhere} \end{cases}$$
$$\Rightarrow \frac{dF}{dy} = \begin{cases} p(y) + p(-y), & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

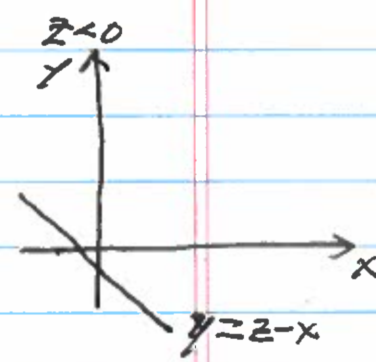
Example:

The joint density of X_1, X_2 is given by

$$p(x,y) = \begin{cases} 6e^{-3x-2y}, & x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of $Z = X + Y$.





$$F(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$
$$= \begin{cases} \int_0^z \int_0^{z-x} 6e^{-3x} e^{-2y}\,dy\,dx, & z > 0 \\ 0 & z < 0 \end{cases}$$
$$= \begin{cases} \int_0^z 3e^{-3x}\left(1 - e^{-2(z-x)}\right)dx, & z > 0 \\ 0, & z < 0 \end{cases}$$
$$= \begin{cases} \int_0^z \left(3e^{-3x} - 3e^{-x-2z}\right)dx, & z > 0 \\ 0, & z < 0 \end{cases}$$

$$\Rightarrow F(z) = \begin{cases} -e^{-3z} + 1 - 3e^{-2z}(1-e^{-z}) , & z > 0 \\ 0 , & z < 0 \end{cases}$$

$$= 1 + 2e^{-3z} - 3e^{-2z}.$$

Therefore, the density is given by

$$g(z) = 1 - 6e^{-3z} + 6e^{-2z}.$$