Lecture 6: Conditional Probability and Independence

Example:

50 car dealerships:

	Good service	Poor Service
≥ 10 year bus.	16	4
< 10 year bus	10	20

Q1:

If somebody randomly selects a dealership what is the probability they get good service?

$$P(G) = \frac{16+10}{50} = .52$$

Q2:

What if they select from businesses with greater than 10 years of experience?

$$P(G|T) = \frac{16}{20} = .8$$

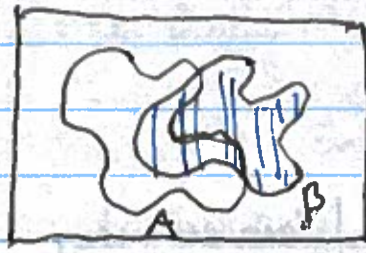
Why does this work?

$$P(G|T) = \frac{|T \cap G|}{|T|} = \frac{|T \cap G|/|S|}{|T|/|S|} = \frac{P(T \cap G)}{P(T)}$$

Definition: If A and B are any two events in a sample space S and $P(A) \neq 0$, the conditional probability of B given A is

$$P(B|A) = P(A \cap B)/P(A).$$

* In words, $P(B|A)$ is the probability of B occuring when restricted to the new sample space A.



Probability of any event $E \subset B$ is simply

$$P_S(E)/P_S(B) = P_B(E)$$

Since $E = A \cap B \subset B$ we typically write this as

$$P(A|B) = P_B(A \cap B) = \frac{P_S(E)}{P_S(B)}.$$

Example:

Loaded die is twice as likely to roll an odd number as an even number.

- What is the probability of a rolling a perfect square?
- What is the probability of rolling a perfect square given the roll is greater than 3.

$S = \{1, 2, 3, 4, 5, 6\}$

Recall $P(\{2\}) = P(\{4\}) = P(\{6\}) = 1/9$, $P(\{1\}) = P(\{3\}) = P(\{5\}) = 2/9$.

Let $E = \{1, 4\}$. Therefore,

$$P(E) = P(\{1,4\}) = P(\{1\} \cup \{4\}) = P(\{1\}) + P(\{4\}) = 3/9 = 1/3.$$

If we let $F = \{4, 5, 6\}$ then

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(\{4\})}{P(\{4,5,6\})} = \frac{P(\{4\})}{P(\{4\}) + P(\{5\}) + P(\{6\})}$$

$$\Rightarrow P(E|F) = \frac{1/9}{4/9} = \frac{1}{4}$$

Theorem - $P(A \cap B) = P(B) \cdot P(A|B)$.

Example:

240 tv's of which 15 are defective.

a) What is probability of getting two defective tv's

b) What is probability of getting only one deffective tv.

c) What is probability of getting no defective tv.

(a) $P(A) = P(\text{tv 1 and tv 2 are defective})$

$= P(\text{tv 2 is defective given 1 is defective})$

$\cdot P(\text{tv 1 is defective})$

$= \frac{14}{239} \cdot \frac{15}{240} = \frac{7}{1912} = .00366$

(b) $\frac{15}{240} \cdot \frac{239-14}{239} + \frac{240-15}{240} \cdot \frac{15}{239} = .1177$

(c) $\frac{240-15}{240} \cdot \frac{239-15}{240} = .8750$

What is probability of getting defective tvs?

$\frac{15}{240} \cdot \frac{14}{239} \cdot \frac{13}{238} = .0002$

Definition - Two events A and B are independent if and only if
$$P(A \cap B) = P(A)\,P(B)$$

Why? If B and A are independent then $P(B|A)$ should equal $P(B)$
$$\Rightarrow P(A \cap B) = P(B|A) \cdot P(A) = P(B) \cdot P(A).$$

Example:
A coin is tossed three times
A = head occurs on first two tosses
B = tails occurs on third toss
C = two tails occur in the three tosses.
$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$\Rightarrow A = \{HHH, HHT\}$, $A \cap B = \{HHT\}$
$B = \{HHT, HTT, THT, TTT\}$, $A \cap C = \emptyset$
$C = \{HTT, THT, TTH\}$ $B \cap C = \{HTT, THT\}$

$\Rightarrow P(A) = \dfrac{|A|}{|S|} = \dfrac{2}{8} = \dfrac{1}{4}$
$P(B) = 1/2$
$P(C) = 3/8$

$P(A \cap B) = 1/8 = P(A)P(B) \Rightarrow$ A and B are independent.
$P(A \cap C) = 0 \neq P(A)P(C) \Rightarrow$ A and C are dependent.
$P(B \cap C) = 1/4 \neq P(B)P(C) \Rightarrow$ B and C are not independent.

Theorem - If A and B are independent, then A and $\bar{B}$ are also independent.

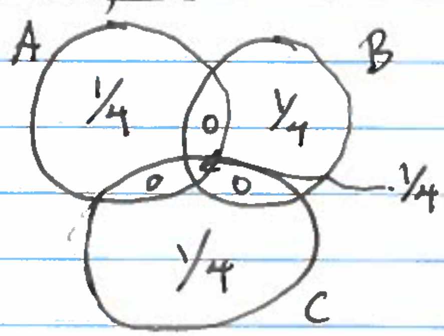
proof:

$$P(A) = P[(A \cap B) \cup (A \cap \bar{B})]$$
$$= P(A \cap B) + P(A \cap \bar{B})$$
$$= P(A)P(B) + P(A \cap \bar{B})$$
$$\Rightarrow P(A) - P(A)P(B) = P(A \cap \bar{B})$$
$$\Rightarrow P(A \cap \bar{B}) = P(A)(1 - P(B))$$
$$= P(A)P(\bar{B}).$$

Definition - Events $A_1, \ldots, A_k$ are independent if

$$P(A_1 \cap A_2 \cdots \cap A_k) = P(A_1)P(A_2)\cdots P(A_k)$$

Example:



$P(A) = 1/2$
$P(B) = 1/2$
$P(C) = 1/2$

$$\left.\begin{aligned} P(A \cap B) &= 1/4 = P(A)P(B) \\ P(A \cap C) &= 1/4 = P(A)P(C) \\ P(B \cap C) &= 1/4 = P(B)P(C) \end{aligned}\right\} \text{independence}$$

$P(A \cap B \cap C) = 1/4 \neq P(A)P(B)P(C) \rightarrow$ Not independent.