MTH 357/657
Quiz #10

1. Suppose X, Y are independent continuous random variables with probability densities

$$p_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases},$$

$$p_Y(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & y < 0 \end{cases},$$

respectively and $Z = (X+Y)/2$, i.e., the average.

(a) Compute $\mathbb{E}[Z]$.

$$\mathbb{E}[Z] = \frac{1}{2}(\mathbb{E}[X] + \mathbb{E}[Y]) = \mathbb{E}[X] = \int_0^\infty x e^{-x}\,dx = \Gamma(2) = 1.$$

(b) Compute the standard deviation of Z.

$$\sigma_Z^2 = \mathrm{Cov}\left(\frac{X+Y}{2}, \frac{X+Y}{2}\right) = \frac{1}{4}[\mathrm{Cov}(X+Y, X+Y)]$$

$$= \frac{1}{4}[\mathrm{Cov}(X,X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,Y)]$$

$$= \frac{1}{4}[\sigma_X^2 + 0 + \sigma_Y^2]$$

$$= \frac{1}{2}\sigma_X^2$$

$$\sigma_X^2 = \int_0^\infty x^2 e^{-x}\,dx - 1 = \Gamma(3) - 1 = 1.$$

$$\Rightarrow \sigma_Z = \frac{1}{\sqrt{2}}.$$