Linear Algebra II
Spring 2024
Exam 2
03/29/24

Name (Print): Key

This exam contains 8 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	5	
3	10	
4	10	
5	15	
6	15	
7	15	
8	10	
9	10	
Total:	100	

1. (10 points) **(Short Answer)** Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If $A \in M_{n\times n}(\mathbb{R})$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvectors $\vec{v}_1, \ldots \vec{v}_n$ corresponding to $\lambda_1, \ldots, \lambda_n$ are all orthogonal with respect to the standard inner product.

   C **I** If two matrices are similar then they have the same eigenvalues.

   C **I** If two matrices are similar then they have the same eigenvectors.

   C **I** If all of the eigenvalues of $A \in M_{n\times n}(\mathbb{R})$ are zero then $A = 0$.

   C **I** If all of the singular values of $A \in M_{n\times n}(\mathbb{R})$ are zero then $A = 0$.

2. (5 points) **Short Answer:** Write down the properties a mapping $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$ must satisfy to be an inner product on a vector space V.

   1. $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$
   2. $\langle a\vec{u}, \vec{v} \rangle = \bar{a}\langle \vec{u}, \vec{v} \rangle$ or $\langle a\vec{u}, \vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle$
   3. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
   4. $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = 0$.

3. (10 points) Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ be orthonormal vectors with respect to the standard inner product and let

$$\vec{u}_1 = 2\vec{v}_1 + 3\vec{v}_2,$$
$$\vec{u}_2 = 2\vec{v}_1 - \vec{v}_2.$$

(a) (5 points) Compute $\|\vec{u}_2\|$.

$$\|U_2\|^2 = \langle 2\vec{v}_1 - \vec{v}_2, 2\vec{v}_1 - \vec{v}_2 \rangle$$
$$= 4\langle \vec{v}_1, \vec{v}_1 \rangle - 4\langle \vec{v}_1, \vec{v}_2 \rangle + \langle \vec{v}_2, \vec{v}_2 \rangle$$
$$= 5$$
$$\Rightarrow \|\vec{v}_2\| = \sqrt{5}.$$

(b) (5 points) Compute $\langle \vec{u}_1, \vec{u}_2 \rangle$.

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \langle 2\vec{v}_1 + 3\vec{v}_2, 2\vec{v}_1 - \vec{v}_2 \rangle$$
$$= 4\|\vec{v}_1\|^2 - 3\|\vec{v}_2\|^2$$
$$= 1$$

4. (10 points) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^4$ be given by

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Find orthonormal vectors $\vec{q}_1, \vec{q}_2, \vec{q}_3 \in \mathbb{R}^4$ such that $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \text{span}\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$.

$$\vec{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \quad \vec{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Let $\vec{w}_3 = \vec{a}_3 - \langle a_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle \vec{q}_2$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - 1/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} - 1/\sqrt{2} \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Since $\|\vec{w}_3\| = 1$, $\vec{q}_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$.

5. (15 points)

(a) (5 points) **Short Answer:** Let $A \in M_{n\times n}(\mathbb{C})$. Write down two equivalent definitions that A must satisfy in order to be a unitary matrix.

- $A^* = A^{-1}$
- The columns of A are orthonormal

(b) (5 points) **Short Answer:** Let $A \in M_{n\times n}(\mathbb{C})$. Write down two equivalent definitions that A must satisfy in order to be a Hermitian matrix.

- $A^* = A$
- For all $\vec{v}, \vec{w} \in \mathbb{C}^n$, $\langle A\vec{v}, \vec{w}\rangle = \langle \vec{v}, A\vec{w}\rangle$.

(c) (5 points) Show that if $A \in M_{n\times n}(\mathbb{C})$ is unitary and $A^2 = I$ then A is Hermitian.

$$A^2 = I$$
$$\Rightarrow A^*A^2 = A^*$$
$$\Rightarrow \cancel{A^*A}\, A = A^*$$
$$\Rightarrow A = A^*.$$

6. (15 points)

(a) (5 points) **Short Answer:** Let U be a subspace of a vector space V. Write down the definition of $U^{\perp}$.

$U^{\perp} = \{\vec{v} \in V : \langle \vec{v}, \vec{u} \rangle = 0 \text{ for all } \vec{u} \in U\}$.

(b) (5 points) Give an example of a matrix $A \in M_{4\times4}(\mathbb{R})$ for which

$$\ker(A) = \operatorname{im}(A).$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) (5 points) Give an example of a matrix $A \in M_{4\times4}(\mathbb{R})$ for which

$$\ker(A) = \operatorname{im}(A)^{\perp}.$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. (15 points) Let $\vec{u} \in M_{n\times n}(\mathbb{C})$ satisfy $\|\vec{u}\| = 1$ with respect to the standard inner product. Let $A \in M_{n\times n}(\mathbb{C})$ be the matrix defined by

$$A = I - 2\vec{u}\vec{u}^*.$$

(a) (5 points) Show that $A^2 = I$.

$$A = I - 2\vec{v}\vec{v}^*$$
$$\Rightarrow A^2 = (I - 2\vec{v}\vec{v}^*)(I - 2\vec{v}\vec{v}^*)$$
$$= I - 4\vec{v}\vec{v}^* + 4\vec{v}\vec{v}^*\vec{v}\vec{v}^*$$
$$= I - 4\vec{v}\vec{v}^* + 4\cdot\|\vec{v}\|^2\vec{v}\vec{v}^*$$
$$= I$$

(b) (5 points) Show that $\vec{u}$ is an eigenvector of A and find its corresponding eigenvalue.

$$A\vec{v} = \vec{v} - 2\vec{v}\vec{v}^*\vec{v}$$
$$= \vec{v} - 2\vec{v}$$
$$= -\vec{v}.$$

Therefore, $\vec{v}$ is an eigenvector with eigenvalue $\lambda = -1$.

(c) (5 points) Suppose $\vec{v}$ is orthogonal to $\vec{u}$. Show that $\vec{v}$ is an eigenvector of A and find its corresponding eigenvalue.

$$A\vec{v} = \vec{v} - 2\vec{u}\vec{u}^*\vec{v}$$
$$= \vec{v}$$

Therefore, $\vec{v}$ is an eigenvector with eigenvalue $\lambda = 1$.

8. (10 points) **Short Answer:** Find the SVD of the following matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\sigma_1 = 3$, $\sigma_2 = 2$, $\sigma_1 = 1$,

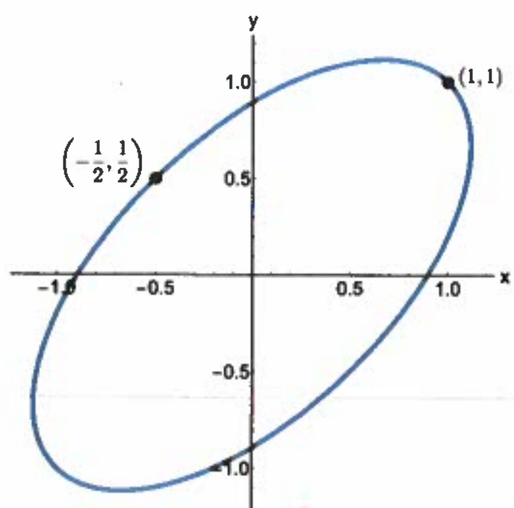
$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\Rightarrow A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

9. (10 points) The figure below is the image of unit ball under the action of a matrix $A \in M_{2\times 2}(\mathbb{C})$. The points on the ellipse are located at the major and semi-major axis.

Short Answer: If the SVD of A is given by $A = U\Sigma V^*$, find possible matrices for U, Σ and V or briefly explain why it is impossible to determine one or more of these matrices.



$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\Rightarrow U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$,

$\vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

V cannot be determined.