MTH 225: Homework #10

Due Date: April 26, 2024

2 1. For each of the following matrices, first determine if it is a transition matrix. If it is a transition matrix, sketch the associated graph along with the edges labeled by their associated probabilities and determine if the matrix is regular. If the matrix is regular, find the stationary probability vector.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{3} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \frac{1}{5} \\ 1 & \frac{4}{5} \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} .3 & .3 & .2 \\ .3 & .2 & .5 \\ .4 & .3 & .3 \end{bmatrix}, \quad G = \begin{bmatrix} .1 & .5 & .4 \\ .6 & .1 & .3 \\ .3 & 0 & .7 \end{bmatrix}, \quad H = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & .2 & 0 & 1 \\ .5 & 0 & .3 & 0 \\ 0 & .8 & 0 & 0 \\ .5 & 0 & .7 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} .1 & .2 & .3 & .4 \\ .2 & .5 & .3 & .1 \\ .3 & .3 & .1 & .3 \\ .4 & .1 & .3 & .2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & .6 & 0 & .4 \\ .5 & 0 & .3 & .1 \\ 0 & .4 & 0 & .5 \\ .5 & 0 & .7 & 0 \end{bmatrix}.$$

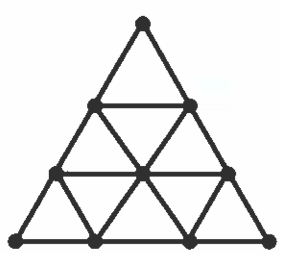
2 2. A business executive is managing three branches, labeled A, B, and C, of a corporation. They never visit the same branch on consecutive days. If they visit branch A one day, they visit branch B the next day. If they visit either branch B or C that day, then the next day the are twice as likely to visit branch A as to visit branch B or C. Explain why the resulting transition matrix is regular. Which branch do they visit the most often in the long run?

3. A traveling salesman visits the three cities of Atlanta, Boston, and Chicago. The matrix

$$T = \begin{bmatrix} 0 & .5 & .5 \\ 1 & 0 & .5 \\ 0 & .5 & 0 \end{bmatrix}$$

describes the transition probabilities of their trip. Describe their travels in words, and calculate how they visit each city on average.

2 4. Suppose an insect crawls along the edges of the graphs drawn below. Upon arriving at a vertex, there is an equal probability of choosing any edge to leave the vertex. For each graph, set up the Markhov chain described by the insect's motion, and determine how often, on average, it visits each vertex.



2 5. Let T be a regular transition matrix with corresponding stationary probability vector $\vec{u}^*$. ,

(a) Prove that $\lim_{k\to\infty} T^k = P = [\vec{u}^*|\vec{u}^*|\cdots|\vec{u}^*]$, i.e., P is the matrix with every column equal to $\vec{u}^*$. **Hint:** This about how you would compute each column of T^k and then take the limit.

(b) Explain why $P\vec{u}^* = \vec{u}^*$.

(c) Prove that P satisfies $P^2 = P$.

(d) Find $\lim\limits_{k\to\infty} T^k$ when

$$T = \begin{bmatrix} .8 & .1 & .1 \\ .1 & .8 & .1 \\ .1 & .1 & .8 \end{bmatrix}$$

6. Prove that, for all $0 \leq p, q \leq 1$ with $p + q > 0$, the stationary probability vector of the transition matrix

$$T = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$$

is $\vec{u}^*$ is $\vec{u}^* = \left[\frac{q}{p+q}, \frac{p}{p+q}\right]^T$.

2 7. Let T be a transition matrix. Prove that if $\vec{u}$ is a probability vector, then so is $\vec{v} = T\vec{u}$.

8. Prove that if T and S are transition matrices, then so is their product TS.

9. Prove that if T is a transition matrix, then so is T^k for all $k \in \mathbb{N}$.

Homework #10

#1.

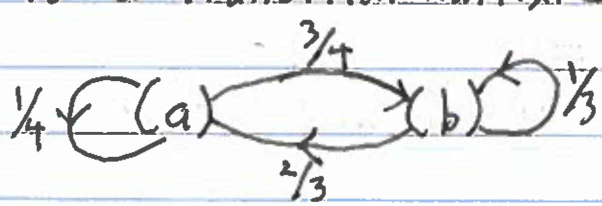
For each of the following matrices, first determine if it is a transition matrix. If it is a transition matrix, sketch the associated graph along with the edges labeled by their associated probabilities and determine if the matrix is regular. If the matrix is regular, find the stationary probability vector.

$$A=\begin{bmatrix} 1/2 & 1/3 \\ 3/4 & 2/3 \end{bmatrix},\quad B=\begin{bmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{bmatrix},\quad C=\begin{bmatrix} 1/4 & 2/3 \\ 3/4 & 1/3 \end{bmatrix},\quad D=\begin{bmatrix} 0 & 1/5 \\ 1 & 4/5 \end{bmatrix}$$

$$H=\begin{bmatrix} 1/2 & 1/2 & 1/3 \\ 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/3 \end{bmatrix},\quad J=\begin{bmatrix} 0 & .2 & 0 & 1 \\ .5 & 0 & .3 & 0 \\ 0 & .8 & 0 & 0 \\ .5 & 0 & .7 & 0 \end{bmatrix},\quad L=\begin{bmatrix} 0 & .6 & 0 & .4 \\ .5 & 0 & .3 & .1 \\ 0 & .4 & 0 & .5 \\ .5 & 0 & .7 & 0 \end{bmatrix}$$

Solution:

- A is not a transition matrix.
- B is not a transition matrix.
- C is a transition matrix. Its graph is



It is clearly regular since C contains only positive entries. In augmented matrix form the equation for the stationary state is

$$\begin{bmatrix} 1/4 & 2/3 \\ 3/4 & 1/3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -3/4 & 2/3 & 0 \\ 3/4 & -2/3 & 0 \end{array}\right]$$

$$\Rightarrow -9v_1 + 8v_2 = 0$$

$$\Rightarrow v_1 = \frac{8}{9} v_2$$

We also know $v_1 + v_2 = 1$ and thus

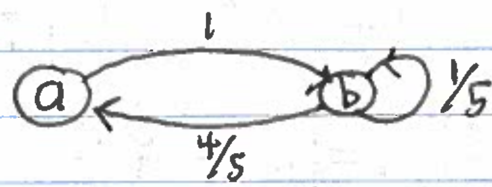
$$\frac{8}{9}v_2 + v_2 = 1$$

$$\Rightarrow v_2 = 9/17$$

$$\Rightarrow v_1 = 8/17$$

$$\Rightarrow \vec{v}^* = \begin{bmatrix} 8/17 \\ 9/17 \end{bmatrix}$$

- D is a transition matrix its graph is



$$D^2 = \begin{bmatrix} 0 & 1/5 \\ 1 & 4/5 \end{bmatrix}\begin{bmatrix} 0 & 1/5 \\ 1 & 4/5 \end{bmatrix} = \begin{bmatrix} 1/5 & 4/25 \\ 4/5 & 21/25 \end{bmatrix}.$$

Consequently, D is regular. The equation for the stationary state is

$$\begin{bmatrix} 0 & 1/5 \\ 1 & 4/5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -1 & 1/5 & 0 \\ 1 & -1/5 & 0 \end{array}\right]$$

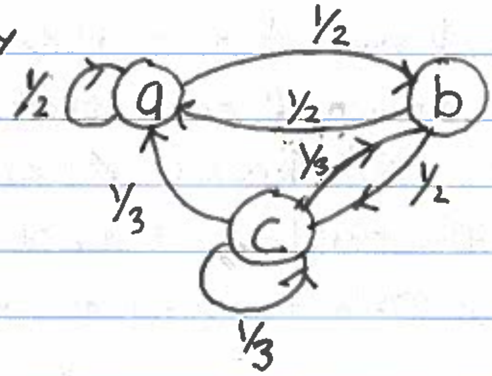
$$\Rightarrow v_1 = 1/5\, v_2 \quad \text{and} \quad v_1 + v_2 = 1$$

$$\Rightarrow v_2 = 5v_1 \Rightarrow 6v_1 = 1 \Rightarrow v_1 = 1/6 \text{ and } v_2 = 5/6$$

Therefore,

$$\vec{v}^* = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$$

- F is not a transition matrix.
- H is a transition matrix and its graph is given by



$$H^2 = \begin{bmatrix} 1/2 & 1/2 & 1/3 \\ 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/3 \\ 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/2 & 5/12 & 4/9 \\ 1/4 & 5/12 & 5/18 \\ 1/4 & 2/12 & 5/18 \end{bmatrix}$$

Therefore, H is regular. The stationary probability satisfies:

$$\begin{bmatrix} 1/2 & 1/2 & 1/3 \\ 1/2 & 0 & 1/3 \\ 0 & 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -1/2 & 1/2 & 1/3 & 0 \\ 1/2 & -1 & 1/3 & 0 \\ 0 & 1/2 & -2/3 & 0 \end{array}\right] \xrightarrow{+R1} \left[\begin{array}{ccc|c} -1/2 & 1/2 & 1/3 & 0 \\ 0 & -1/2 & 2/3 & 0 \\ 0 & 1/2 & -2/3 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1/2 & 1/2 & 3 & 0 \\ 0 & -1/2 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore,

$$v_2 = 4/3\, v_3$$
$$-1/2\, v_1 + 4/3\, v_3 + 3 v_3 = 0$$
$$1/2\, v_1 = 13/3\, v_3$$
$$v_1 = \frac{26}{3} v_3$$

We also have

$$v_1 + v_2 + v_3 = 1$$
$$\Rightarrow \frac{26}{3} v_3 + \frac{4}{3} v_3 + v_3 = 1$$
$$\Rightarrow 33 v_3 = 3$$
$$v_3 = 1/11$$

$$\Rightarrow v_2 = \frac{4}{33}$$
$$v_1 = \frac{26}{33}$$

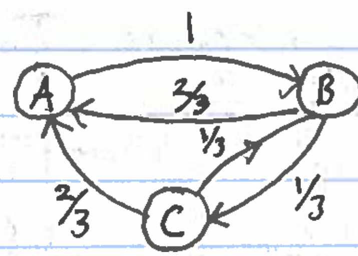
$$\Rightarrow \vec{v}^* = \begin{bmatrix} 26/33 \\ 4/33 \\ 1/11 \end{bmatrix}$$

#2.

A business executive is managing three branches, labeled A, B, and C, of a corporation. If they visit branch A then they visit branch B the next day. If they visit branch B or C then the next day they are twice as likely to visit branch A as to visit branch B or C. Explain why the resulting transition matrix is regular. Which branch do they visit the most often in the long run.

<u>Solution:</u>

$$T = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2/3 & 2/3 \\ 1 & 0 & 1/3 \\ 0 & 1/3 & 0 \end{bmatrix} \end{array}$$



The stationary state satisfies

$$\begin{bmatrix} 0 & 2/3 & 2/3 \\ 1 & 0 & 1/3 \\ 0 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2/3 & -2/3 & 0 \\ -1 & 1 & -1/3 & 0 \\ 0 & -1/3 & 1 & 0 \end{array}\right] \begin{matrix} \\ +R1 \\ \\ \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2/3 & -2/3 & 0 \\ 0 & 1/3 & -1 & 0 \\ 0 & -1/3 & 1 & 0 \end{array}\right]$$

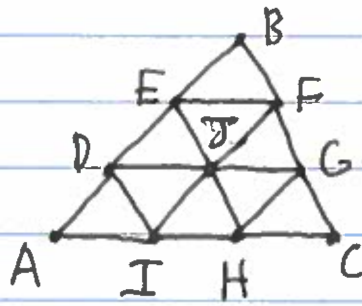
$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2/3 & -2/3 & 0 \\ 0 & 1/3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow u_2 = 3u_3,\ u_1 - 2u_3 - 2/3\, u_3 = 0 \Rightarrow u_1 = 8/3\, u_3.$$

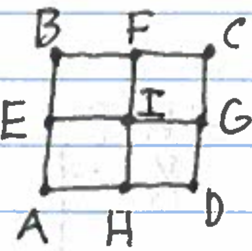
Since $u_2 = 3u_3$ and $u_1 = 8/3\, u_3$ it follows that they visit branch B the most.

■

#4.

Suppose an insect crawls along the edges of the graphs drawn below. Upon arriving at a vertex, there is an equal probability of choosing any edge to leave the vertex. For each graph, set up the Markhov chain described by the insect's motion, and determine how often, on average, it visits each vertex.





The first matrix is

$$\begin{array}{c|cccccccccc}
 & A & B & C & D & E & F & G & H & I & J \\
\hline
A & & & & 1/4 & & & & & 1/4 & \\
B & & & & & 1/4 & 1/4 & & & & \\
C & & & & & & & 1/4 & 1/4 & & \\
D & 1/2 & & & & & & & & 1/4 & 1/6 \\
E & & 1/2 & & 1/4 & 1/4 & 1/4 & & & & 1/6 \\
F & & 1/2 & & & 1/4 & & 1/4 & & & 1/6 \\
G & & & 1/2 & & & 1/4 & & 1/4 & & 1/6 \\
H & & & 1/2 & & & & 1/4 & & 1/4 & 1/6 \\
I & 1/2 & & & 1/4 & & & & 1/4 & & 1/6 \\
J & & & & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 &
\end{array}$$

*Blank entries are zero.

If we let p denote the probability of being at a corner we have $3p + 6 \cdot 2p + 3p = 1$

Consequently, $p = 1/18$ and thus we obtain the following stationary probability

$$\vec{v}^* = \frac{1}{18}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

The second matrix is

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} A & B & C & D & E & F & G & H & I \end{array} \\
\left[\begin{array}{ccccccccc}
 & & & & 1/3 & & & 1/3 & \\
 & & & & 1/3 & 1/3 & & & \\
 & & & & & 1/3 & 1/3 & & \\
 & & & & & & 1/3 & 1/3 & \\
1/2 & 1/2 & & & & & & & 1/4 \\
 & 1/2 & 1/2 & & & & & & 1/4 \\
 & & 1/2 & 1/2 & & & & & 1/4 \\
1/2 & & & 1/2 & & & & & 1/4 \\
 & & & & 1/3 & 1/3 & 1/3 & 1/3 &
\end{array}\right]
\end{array}$$

* Blank entries are zero

If we let p denote the probability of being at a corner we have that

$$4p + 4\cdot\left(\tfrac{3}{2}\right)p + 2p = 1 \Rightarrow \vec{v}^* = \frac{1}{24}\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 3 \\ 3 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow 4p + 6p + 2p = 1$$

$$\Rightarrow p = 1/12$$

#5

Let T be a regular transition matrix with stationary probability vector $\vec{v}^*$.

(a) Prove that $\lim_{k\to\infty} T^k = P = [\vec{v}^* | \vec{v}^* | \cdots | \vec{v}^*]$.

(b) Explain why $P\vec{v}^* = \vec{v}^*$.

(c) Prove that $P^2 = P$.

(d) Find $\lim_{k\to\infty} T^k$ when

$$T = \begin{bmatrix} .8 & .1 & .1 \\ .1 & .8 & .1 \\ .1 & .1 & .8 \end{bmatrix}$$

Solution:

(a) Let $\{\vec{e}_1, \dots, \vec{e}_n\}$ denote the standard basis vectors which are all probability vectors. Consequently,

$$\lim_{k\to\infty} T^k \vec{e}_i = \vec{v}^*.$$

Therefore,

$$\lim_{k\to\infty} T^k = \lim_{k\to\infty} [T^k\vec{e}_1 | T^k\vec{e}_2 | \cdots | T^k\vec{e}_n]$$

$$= [\vec{v}^* | \vec{v}^* | \cdots | \vec{v}^*].$$

(b) Since $\vec{v}^*$ is stationary we have for all k that

$$T^k\vec{v}^* = \vec{v}^*$$

$$\Rightarrow \lim_{k\to\infty} T^k \vec{v}^* = \vec{v}^*$$

$$\Rightarrow P\vec{v}^* = \vec{v}^*.$$

(c) $$P^2 = P[\vec{v}^* | \cdots | \vec{v}^*]$$

$$= [P\vec{v}^* | \cdots | P\vec{v}^*]$$

$$= [\vec{v}^* | \cdots | \vec{v}^*]$$

$$= P.$$

(d) We need to find the stationary probability vector of T.

$$\Rightarrow \left[\begin{array}{ccc|c} .2 & -.1 & -.1 & 0 \\ -.1 & .2 & -.1 & 0 \\ -.1 & -.1 & .2 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{matrix} /2 \\ /2 \\ /2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix} \begin{matrix} \\ +\frac{1}{2}R1 \\ +\frac{1}{2}R1 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 3/4 & -3/4 & 0 \\ 0 & -3/4 & 7/4 & 0 \end{array}\right]$$

$\Rightarrow u_2 = u_3 = u_1$ (I should have guessed this from symmetry).

Therefore, $\vec{u}^* = 1/3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Consequently

$$\lim_{k \to \infty} T^k = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\blacksquare$

#7

Let T be a transition matrix. Prove that if $\vec{u}$ is a probability vector then so is $\vec{v} = T\vec{u}$.

Solution:

Let $\vec{v} = [v_1, \ldots, v_n]^T$ and $\vec{u} = [u_1, \ldots, u_n]^T$. Therefore,

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} u_j$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} T_{ij} u_j$$

$$= \sum_{j=1}^{n} u_j \sum_{i=1}^{n} T_{ij}$$

$$= \sum_{j=1}^{n} u_j \cdot 1$$

$$= 1.$$

$\blacksquare$

#8.

Prove that if T and S are transition matrices then so is there product TS.

proof

Let $U = TS$ and U_{ij}, T_{ij}, S_{ij} denote the components of each matrix respectively. Therefore,

$$U_{ij} = \sum_{k=1}^{n} T_{ik} S_{kj}$$

$$\Rightarrow \sum_{i=1}^{n} U_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{n} T_{ik} S_{kj}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} T_{ik} S_{kj}$$

$$= \sum_{k=1}^{n} S_{kj} \sum_{i=1}^{n} T_{ik}$$

$$= \sum_{k=1}^{n} S_{kj} \cdot 1$$

$$= 1.$$

$\blacksquare$