MTH 225: Homework #1

Due Date: January 26, 2024

1. Sign up for Piazza. I will check the roster for your name.

2. Solve the following systems of equations using row reduction. For those systems with infinitely many solutions, find a parametric description of the solution space.

(a)

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v + w &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

(b)

$$\begin{aligned} 2x + 3y &= 7 \\ x - 3y &= 5 \\ 5x + 3y &= 18 \end{aligned}$$

(c)

$$\begin{aligned} x - z &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

3. Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$ under the standard operations of vector addition and scalar multiplication. If a set is a subspace, prove it. If a set is not a subspace explain why.

   (a) $V = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$
   (b) $W = \{(x, y, z) \in \mathbb{R}^3 : y = 1\}$
   (c) $U = \{(x, y, z) \in \mathbb{R}^3 : xy = 0\}$
   (d) $S = \{(x, y, z) \in \mathbb{R}^3 : z - x = 2y\}$

4. Show that the set of 2×2 matrices of the following form is a subspace of $M_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} \text{ for } a, b, c \in \mathbb{R}.$$

5. Determine whether the following sets are subspaces of $P_2(\mathbb{R})$. If a set is a subspace prove it. If a set is not a subspace explain why it is not.

   (a) Polynomials of the form $p(t) = at^2$ for $a \in \mathbb{R}$.
   (b) Polynomials of the form $p(t) = a + t^2$ for $a \in \mathbb{R}$.
   (c) Polynomials $p(t)$ in $P_2(\mathbb{R})$ with integer coefficients.
   (d) Polynomials $p(t)$ in $P_2(\mathbb{R})$ with $p(0) = 0$.

6. Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?

7. Find a vector that, together with the vectors $[1, 1, 1]^T$ and $[1, 2, 1]^T$, forms a basis of $\mathbb{R}^3$.

8. Determine if the following sets of vectors are linearly independent and determine a basis for its span.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 11 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

9. Suppose that W_1 and W_2 are subspaces of $\mathbb{R}^n$. Define the following subsets of $\mathbb{R}^n$:

$$\begin{aligned} W_1 + W_2 &= \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\} \\ W_1 \cap W_2 &= \{w \in \mathbb{R}^n \mid w \in W_1 \text{ and } w \in W_2\} \\ W_1 \cup W_2 &= \{w \in \mathbb{R}^n \mid w \in W_1 \text{ or } w \in W_2\} \end{aligned}$$

(a) Prove that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

(b) Prove that $W_1 + W_2$ is a subspace of $\mathbb{R}^n$.

(c) Show by means of an example that $W_1 \cup W_2$ is not necessarily a subspace of $\mathbb{R}^n$. Which of the three rules are not satisfied?

(d) Show that W_1 and W_2 are subsets of $W_1 + W_2$.

(e) Show that if V is another subspace that contains both W_1 and W_2, then V contains $W_1 + W_2$.

Homework #1

#4

Show that the set of 2x2 matrices of the following form is a subspace of $M_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix}, \quad a, b, c \in \mathbb{R}$$

Solution:

Let $M_1 = \begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix}$, $M_2 = \begin{bmatrix} a_2 & b_2 \\ -b_2 & c_2 \end{bmatrix}$. Therefore,

$$M_1 + M_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ -b_1-b_2 & c_1+c_2 \end{bmatrix},$$

$$\lambda M_1 = \begin{bmatrix} \lambda a_1 & \lambda b_1 \\ -\lambda b_1 & \lambda c_1 \end{bmatrix}$$

Consequently, this set of matrices is a subspace of $M_{2\times 2}(\mathbb{R})$. ∎

#5

Determine whether the following sets are subspaces of $P_2(\mathbb{R})$. If a set is a subspace prove it. If a set is not a subspace explain why it is not.

(a) Polynomials of the form $p(t) = at^2$; $a \in \mathbb{R}$

(b) Polynomials of the form $p(t) = a + t^2$, $a \in \mathbb{R}$

(c) Polynomials $p(t)$ with integer coefficients.

(d) Polynomials with $p(0) = 0$.

Solution:

(a) Let $p_1(t) = a_1 t^2$, $p_2(t) = a_2 t^2$. Therefore,

$$p_1(t) + p_2(t) = a_1 t^2 + a_2 t^2 = (a_1 + a_2)t^2,$$

$$c p_1(t) = c a_1 t^2,$$

and thus this set is a subspace.

(b) This set is not a Subspace since $1+t^2$ is in the set but $3(1+t^2)=3+3t^2$ is not.

(c) This is not a subspace since t^2 is in the set but $\frac{3}{2}t^2$ is not.

(d) Let p_1, p_2 be in this set. Therefore,

$$(p_1+p_2)(0)=p_1(0)+p_2(0)=0+0=0$$
$$cp_1(0)=c\cdot 0=0$$

Consequently, this set is a subspace. $\blacksquare$

#6

Are the functions $1+x$, $1-x$, and $1+x+x^2$ linearly dependent or independent? Why?

Solution:

If $1+x$, $1-x$, and $1+x+x^2$ are linearly independent the only constants which satisfy

$$c_1(1+x)+c_2(1-x)+c_3(1+x+x^2)=0$$

are the constants $c_1=c_2=c_3=0$. Therefore,

$$c_1+c_2+c_3+(c_1-c_2)x+c_3x^2=0.$$

Therefore,

$$\begin{aligned} c_1+c_2+c_3&=0\\ c_1-c_2\quad\;&=0\\ c_3&=0 \end{aligned}$$

$$\Rightarrow c_3=0,\; c_1=c_2,\; c_1=-c_2.$$

The only solution is $c_1=c_2=c_3=0$ and thus these functions are linearly independent.

#8

Determine if the following sets of vectors are linearly independent and determine a basis for its span.

(a) $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix} \right\}$, (c) $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\11\end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix} \right\}$, (f) $\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

Solution:

(a) Row reducing, we have that

$$\begin{bmatrix}1&2&3\\2&3&2\\3&1&1\end{bmatrix}\begin{matrix}\\-2R2\\-3R1\end{matrix} \Rightarrow \begin{bmatrix}1&2&3\\0&-1&-4\\0&-5&-8\end{bmatrix}\begin{matrix}+2R2\\ \\-5R2\end{matrix} \Rightarrow \begin{bmatrix}1&0&-5\\0&-1&-4\\0&0&12\end{bmatrix}$$

Therefore the vectors are linearly independent and thus form a basis for there own span.

(c) Row reducing, we have that

$$\begin{bmatrix}1&1&1\\2&2&2\\3&1&11\end{bmatrix}\begin{matrix}\\-2R1\\-3R1\end{matrix} \Rightarrow \begin{bmatrix}1&1&1\\0&0&0\\0&-2&8\end{bmatrix}\begin{matrix}\\ \\+2R1\end{matrix} \Rightarrow \begin{bmatrix}1&1&1\\0&0&0\\0&0&10\end{bmatrix}$$

Therefore, the set is linearly dependent with the basis being

$$\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\11\end{bmatrix} \right\}$$

(d) Row reducing, we have that

$$\begin{bmatrix}1&1&3\\1&2&2\\1&3&1\end{bmatrix}\begin{matrix}\\-R1\\-R1\end{matrix} \Rightarrow \begin{bmatrix}1&1&3\\0&1&-1\\0&2&-2\end{bmatrix}\begin{matrix}\\ \\-R2\end{matrix} \Rightarrow \begin{bmatrix}1&1&3\\0&1&-1\\0&0&0\end{bmatrix}$$

Therefore, the set is linearly dependent with the basis being

$$\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}$$

(f) Row reducing we have that

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{matrix} \\ -2R1 \\ -R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{matrix} \\ \\ +2R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the set is linearly dependent with the basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

#9

Suppose that W_1, W_2 are subspaces of $\mathbb{R}^n$.

(a) Prove that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

(b) Prove that $W_1 + W_2$ is a subspace of $\mathbb{R}^n$.

(c) Show by means of an example that $W_1 \cup W_2$ is not neccessarilly a subspace of $\mathbb{R}^n$.

(d) Show that W_1 and W_2 are subsets of $W_1 + W_2$.

(e) Show that if V is another subspace that contains both W_1 and W_2 then V contains $W_1 + W_2$.

Solution:

(a) Let $v_1, v_2 \in W_1 \cap W_2$. Therefore, $v_1, v_2 \in W_1, W_2$ and thus $v_1 + v_2 \in W_1, W_2$ proving that $v_1 + v_2 \in W_1 \cap W_2$. Moreover, for $\lambda \in \mathbb{R}$ we have that $\lambda v_1 \in W_1, W_2$ proving that $\lambda v_1 \in W_1 \cap W_2$. By these two properties it follows that $W_1 \cap W_2$ is a subspace.

(b) Let $v_1, v_2 \in W_1 + W_2$. Consequently, there exists $s_1, s_2 \in W_1$, $t_1, t_2 \in W_2$ such that $v_1 = s_1 + t_1$ and $v_2 = s_2 + t_2$. Therefore, since $s_1 + s_2 \in W_1$ and $t_1 + t_2 \in W_2$, it follows that $v_1 + v_2 \in W_1 + W_2$ since

$$v_1 + v_2 = (s_1 + s_2) + (t_1 + t_2).$$

For $\lambda \in \mathbb{R}$ it follows that

$$\lambda v_1 = \lambda (s_1 + t_1) = \lambda s_1 + \lambda t_1.$$

Therefore, $\lambda v_1 \in W_1 + W_2$ since $\lambda s_1 \in W_1$ and $\lambda t_1 \in W_2$.

(c) Let $W_1 = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ and $W_2 = \text{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. Therefore, $\begin{bmatrix}1\\0\\0\end{bmatrix} \in W_1$ and $\begin{bmatrix}0\\1\\0\end{bmatrix} \in W_2$ but $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ which is not in either W_1 or W_2. Consequently, $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} \notin W_1 \cup W_2$

(d) For all $v \in W_1$, since $0 \in W_2$ it follows that

$$v = v + 0$$

and thus $v \in W_1 + W_2$. Consequently, $W_1 \subset W_1 + W_2$. A similar argument shows that $W_2 \subset W_1 + W_2$.

(e) Let V be a subspace containing W_1 and W_2. Now let $v \in W_1 + W_2$. Consequently, there exists $v_1 \in W_1$, $v_2 \in W_2$ such that $v = v_1 + v_2$. However, since $W_1 \subset V$ and $W_2 \subset V$ it follows that $v_1, v_2 \in V$ and thus $v_1 + v_2 \in V$.

$\blacksquare$