MTH 225: Homework #2

Due Date: February 02, 2024

1. Let $F(a,b)$ be the vector space of all real valued function on the interval (a,b) over the field of real numbers with the normal operations of addition and scalar multiplication. Determine if the following subsets of $F(a,b)$ are a subspace of $F(a,b)$. If they are not a subspace explain why and if they are a subspace prove it.

   (a) All functions f in $F(a,b)$ for which $f(a)=0$.

   (b) All functions f in $F(a,b)$ for which $f(a)=1$.

   (c) All continuous functions f in $F(a,b)$ for which $\int_a^b f(x)dx = 0$.

   (d) All differentiable functions f in $F(a,b)$ for which $f'(x)=f(x)$.

2. Show that
$$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right\}$$
forms a basis for $M_{2\times 2}(\mathbb{R})$.

3. Consider the map $T : P_3(\mathbb{R}) \mapsto \mathbb{R}^4$ defined by
$$T(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_1, a_1, a_1, a_1).$$

   (a) Show that T is a linear transformation.

   (b) Find bases for $\ker(T)$, $\mathrm{im}(T)$ and verify the Rank + Nullity Theorem.

4. Explain why there is no linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ that satisfies the following
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 15 \\ 6 \\ 5 \end{bmatrix}.$$

5. Define the linear transformation $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{R}^3$ by
$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a+b \\ b+c \\ a+b \end{bmatrix}.$$

   (a) Find a basis for the kernel of T.

   (b) Find a basis for the range of T.

6. The trace of an $n \times n$ matrix is the sum of the diagonal entries:

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

(a) Show that $T : M_{3\times 3}(\mathbb{R}) \mapsto \mathbb{R}$ defined by $T(A) = \text{trace}(A)$ is a linear transformation.

(b) Find the dimension and a basis for $\ker(T)$ and generalize to $n \times n$ matrices.

7. Let $A = (a_{ij})$ be an $n \times n$ matrix whose i, j th entry is a_{ij}. Define the transpose of A as the matrix whose ij entry is a_{ji}, i.e. $A^t = (a_{ji})$. A matrix is called *symmetric* if $A = A^t$.

(a) Prove that the symmetric $n \times n$ matrices form a subspace of $M_{n\times n}(\mathbb{R})$.

(b) Find the dimension of the subspace of symmetric $n \times n$ matrices. Prove your answer.

Homework #2

#1

Determine if the following subsets of $F(a,b)$ are a subspace of $F(a,b)$. If they are not explain why and if they are prove it.

(a) All $f \in F(a,b)$ satisfying $f(a)=0$

(b) All $f \in F(a,b)$ satisfying $f(a)=1$

(c) All continuous functions $f \in F(a,b)$ for which $\int_a^b f(x)dx = 0$.

(d) All differentiable $f \in F(a,b)$ satisfying $f'(x)=f(x)$.

Solution:

(a) This is a subspace.

~~proof~~

(i) Let $f, g \in F(a,b)$ satisfy $f(a)=g(a)=0$ and let $h(x)=f(x)+g(x)$. Therefore, $h(a)=f(a)+g(a)=0$.

(ii) Let $f \in F(a,b)$ satisfy $f(a)=0$ and $\lambda \in \mathbb{R}$. If $h(x)=\lambda f(x)$ it follows that $h(a)=\lambda f(a)=0$.

By items (i)-(ii), it follows that this space is a subspace.

(b) This is not a subspace since 0 is not in the set.

(c) This is a subspace.

~~proof~~:

(i) Let $f, g \in F(a,b)$ satisfy $\int_a^b f(x)dx = \int_a^b g(x)dx = 0$ and let $h(x)=f(x)+g(x)$. Therefore,

$$\int_a^b h(x)dx = \int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx = 0+0=0.$$

(ii) Let $f \in F(a,b)$ satisfy $\int_a^b f(x)dx = 0$ and let $\lambda \in \mathbb{R}$. If $h(x)=\lambda f(x)$ it follows that

$$\int_a^b h(x)dx = \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx = 0.$$

By items (i)-(ii) it follows that this is a subspace.

(d) (i) Let $f, g \in F(a,b)$ satisfy $f'(x)=f(x)$ and $g'(x)=g(x)$. Define $h(x)=f(x)+g(x)$. Therefore,

$$h'(x)=(f+g)'(x)=f'(x)+g'(x)=f(x)+g(x)=h(x).$$

(ii) Let $f\in F(a,b)$ satisfy $f'(x)=f(x)$ and let $\lambda\in\mathbb{R}$. Define $h(x)=\lambda f(x)$. Therefore,

$$h'(x)=(\lambda f(x))'=\lambda f'(x)=\lambda f(x)=h(x).$$

By items (i)-(ii) this is a vector space.

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#3.

Consider the map $T: P_3(\mathbb{R})\to\mathbb{R}^4$ defined by

$$T(a_3x^3+a_2x^2+a_1x+a_0)=(a_1,a_1,a_1,a_1)$$

(a) Show that T is a linear transformation.

(b) Find bases for $\ker(T)$, $\operatorname{im}(T)$ and verify the rank nullity theorem.

Solution:

(a) Let $\vec{u},\vec{v}\in P_3(\mathbb{R})$ and $\lambda\in\mathbb{R}$. Therefore, there exists $a_0,a_1,a_2,a_3,b_0,b_1,b_2,b_3\in\mathbb{R}$ such that $\vec{u}=a_3x^3+a_2x^2+a_1x+a_0$, $\vec{v}=b_3x^3+b_2x^2+b_1x+b_0$. Consequently,

(i)
$$\begin{aligned}
T[\vec{u}+\vec{v}]&=T[a_3x^3+a_2x^2+a_1x+a_0+b_3x^3+b_2x^2+b_1x+b_0]\\
&=T[(a_3+b_3)x^3+(a_2+b_2)x^2+(a_1+b_1)x+a_0+b_0]\\
&=(a_0+b_0,a_0+b_0,a_0+b_0,a_0+b_0)\\
&=(a_0,a_0,a_0,a_0)+(b_0,b_0,b_0,b_0)\\
&=T[\vec{u}]+T[\vec{v}].
\end{aligned}$$

(ii)
$$\begin{aligned}
T[\lambda\vec{u}]&=T[\lambda a_3x^3+\lambda a_2x^2+\lambda a_1x+\lambda a_0]\\
&=(\lambda a_1,\lambda a_1,\lambda a_1,\lambda a_1).\\
&=\lambda(a_1,a_1,a_1,a_1)\\
&=\lambda T[\vec{u}].
\end{aligned}$$

By items (i)-(ii), T is a linear transformation.

(b) The basis for $\ker(T)$ is given by

$$\ker(T) = \operatorname{span}\{x^3, x^2, 1\}$$

and the basis for $\operatorname{im}(T)$ is given by

$$\operatorname{im}(T) = \operatorname{span}\{(1,1,1,1)\}.$$

Therefore,

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = 3 + 1 = 4 = \dim(P_3(\mathbb{R})).$$

$\blacksquare$

#4

Explain why there is no linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ that satisfies the following

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\2\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}3\\6\end{bmatrix}\right) = \begin{bmatrix}15\\6\\5\end{bmatrix}.$$

<u>Solution:</u>

Since $\begin{bmatrix}3\\6\end{bmatrix} = 6\begin{bmatrix}1\\1\end{bmatrix} - 3\begin{bmatrix}1\\0\end{bmatrix}$ it follows that we should have

$$\begin{aligned} T\left(\begin{bmatrix}3\\6\end{bmatrix}\right) &= 6T\begin{bmatrix}1\\1\end{bmatrix} - 3T\begin{bmatrix}1\\0\end{bmatrix} \\ &= \begin{bmatrix}18\\12\\6\end{bmatrix} - \begin{bmatrix}3\\6\\9\end{bmatrix} \\ &= \begin{bmatrix}15\\6\\-3\end{bmatrix}, \end{aligned}$$

which is a contradiction.

$\blacksquare$

#5.

Define the linear transformation $T: P_3(\mathbb{R}) \to \mathbb{R}^3$ by

$$T(a+bx+cx^2+dx^3) = \begin{bmatrix} a+b \\ b+c \\ a+b \end{bmatrix}.$$

(a) Find a basis for $\ker(T)$.

(b) Find a basis for the range of T.

Solution:

(a) Suppose $T(a+bx+cx^2+dx^3)=0$. Therefore,

$$\begin{matrix} a+b=0 \\ b+c=0 \\ a+b=0 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is therefore

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right] -R1 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow b=-c,\ a=-b=c$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c \\ -c \\ c \\ d \end{bmatrix} = c\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\ker(T) = \text{span}\{1-x+x^2, x^3\}.$$

(b) Since $\begin{bmatrix} a+b \\ b+c \\ a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ it follows that

$$\text{im}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

To find a basis, we need to determine which of these vectors is linearly dependent. Row reducing, we have that

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} -R1 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the basis is $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

#6.

The trace of an $n\times n$ matrix is defined by

$$\text{trace}(A)=\sum_{i=1}^{n} a_{ii}.$$

(a) Show that $T: M_{3\times3}(\mathbb{R})\to\mathbb{R}$ defined by $T(A)=\text{trace}(A)$ is a linear transformation.

(b) Find the dimension and a basis for $\ker(T)$ and generalize to $n\times n$ matrices.

Solution:

(a) Let $A,B\in M_{3\times3}(\mathbb{R})$ and $\lambda\in\mathbb{R}$. Let a_{ij}, b_{ij} denote the components of A and B.

(i)
$$\begin{aligned}T(A+B)&=\text{trace}(A+B)\\&=\sum_{i=1}^{n}(a_{ii}+b_{ii})\\&=\sum_{i=1}^{n}a_{ii}+\sum_{i=1}^{n}b_{ii}\\&=\text{trace}(A)+\text{trace}(B)\\&=T(A)+T(B)\end{aligned}$$

(ii)
$$\begin{aligned}T(\lambda A)&=\text{trace}(\lambda A)\\&=\sum_{i=1}^{n}\lambda a_{ii}\\&=\lambda\sum_{i=1}^{n}a_{ii}\\&=\lambda\,\text{trace}(A)\\&=\lambda T(A).\end{aligned}$$

By items (i)-(ii), T is a linear transformation.

(b). If $A\in\ker(T)$ then $a_{33}=-a_{22}-a_{11}$ and thus

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&-a_{22}-a_{11}\end{bmatrix}=a_{11}\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}+a_{12}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}+a_{13}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$+a_{21}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}+a_{22}\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}+a_{23}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$+a_{31}\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}+a_{32}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}.$$

Consequently,

$$\ker(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

$$\Rightarrow \dim(\ker(T)) = 8.$$

In general $\dim(\ker(T)) = n^2 - 1$. ∎