# MTH 225: Homework #3

Due Date: February 09, 2024

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with values

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix},\quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\1\end{bmatrix},\text{ and } T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\end{bmatrix}).$$

   (a) Find the matrix $[T(\mathcal{B}, \mathcal{S}_2)]$ of $T$ in the bases

$$\mathcal{B} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \text{ and } \mathcal{S}_2 = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

   where $[T(\mathcal{B}, \mathcal{S}_2)][\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{S}_2}$.

   (b) Find the matrix $[T(\mathcal{S}_1, \mathcal{S}_2)]$ of $T$ in the standard bases

$$\mathcal{S}_1 = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \text{ and } \mathcal{S}_2 = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\},$$

   where $[T(\mathcal{S}_1, \mathcal{S}_2)][\mathbf{v}]_{\mathcal{S}_1} = [T(\mathbf{v})]_{\mathcal{S}_2}$.

2. Consider the two bases $\mathcal{B}$ and $\mathcal{S}$ of $\mathbb{R}^2$ where

$$\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\} \text{ and } \mathcal{S} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

   Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x - 2y\\3x + y\end{bmatrix}.$$

   (a) Find the matrix representation of the identity linear transformation $I : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with respect to the input basis $\mathcal{B}$ and output basis $\mathcal{S}$. That is, find the matrix $P = [I(\mathcal{B}, \mathcal{S})]$ where $P[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{S}}$.

   (b) Find the matrix representation of the identity linear transformation $I : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with respect to the input basis $\mathcal{S}$ and output basis $\mathcal{B}$. That is, find the matrix $Q = [I(\mathcal{S}, \mathcal{B})]$ where $Q[\mathbf{v}]_{\mathcal{S}} = [\mathbf{v}]_{\mathcal{B}}$.

   (c) Find the matrix $A = [T(\mathcal{S}, \mathcal{S})]$, where $A[\mathbf{v}]_{\mathcal{S}} = [T(\mathbf{v})]_{\mathcal{S}}$.

   (d) Find the matrix $B = [T(\mathcal{S}, \mathcal{B})]$, where $B[\mathbf{v}]_{\mathcal{S}} = [T(\mathbf{v})]_{\mathcal{B}}$.

   (e) Write $B$ as a product of $A$ and any of the matrices $P$ and $Q$ that are relevant.

3. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(a + bx + cx^2) = (a + c) + bx^2$. Let $\mathcal{S} = \{1, x, x^2\}$ be the standard basis of $P_2(\mathbb{R})$, and let $\mathcal{B} = \{1 + x, x, 1 + x^2\}$ be another basis.

   (a) Find the matrix $[T(\mathcal{S}, \mathcal{S})]$ of $T$ with respect to $\mathcal{S}$.

   (b) Find the matrix $[T(\mathcal{B}, \mathcal{B})]$ of $T$ with respect to $\mathcal{B}$.

   (c) Find an invertible matrix $P$ so that $P[T(\mathcal{B}, \mathcal{B})]P^{-1} = [T(\mathcal{S}, \mathcal{S})]$.

4. Suppose $\mathbf{v} \in V$ is a vector in some vector space $V$ and $T : V \to V$ a linear transformation such that $\mathcal{B} = \{\mathbf{v}, T(\mathbf{v}), T^2(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is a basis for $V$.

   (a) Show that there exists constants $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ such that

   $$T^n(\mathbf{v}) = a_0\mathbf{v} + a_1T(\mathbf{v}) + \ldots + a_{n-1}T^{n-1}(\mathbf{v}).$$

   (b) Find the matrix $[T(\mathcal{B}, \mathcal{B})]$ of $T$ with respect to $\mathcal{B}$.

   (c) When does $T$ map onto $V$ and when is $T$ one-to-one?

   (d) Find the characteristic polynomial $c_T(x)$ of $T$. (Hint: Do cases $n = 2$ and 3 to get a conjecture and prove it by induction).

5. Suppose that the vector $v = (1, 1) \in \mathbb{R}^2$ is an eigenvector of $A \in \mathrm{Mat}_{2\times2}(\mathbb{R})$ corresponding to the eigenvector $\lambda$. Draw on a graph the vectors $v$ and $Av$ for each of the following cases. (Make a separate graph for each part.)

   (a) $\lambda > 1$.

   (b) $\lambda = 1$.

   (c) $0 < \lambda < 1$.

   (d) $\lambda = 0$.

   (e) $-1 < \lambda < 0$.

   (f) $\lambda = -1$.

   (g) $\lambda < -1$.

6. Prove that 0 is an eigenvalue of $T$ if and only if $Ker(T)$, the nullspace of $T$, is $\neq \{0\}$.

7. If $\lambda$ is an eigenvalue for an $n \times n$ matrix $A$, show that $\lambda^2$ is an eigenvalue for $A^2$. More generally prove that if $f(x)$ is any polynomial with coefficients in $\mathbb{R}$, then $f(\lambda)$ is an eigenvalue for $f(A)$.

8. A matrix $N$ is called *nilpotent* if $N^k = 0$ for some positive integer $k$. Prove that the only possible eigenvalue of a nilpotent matrix is 0.

Homework #3

#1.

Solution:

(a) Since $T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}-2\\1\end{bmatrix}$, $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\2\end{bmatrix}$

it follows that

$$[T(\beta, S_2)]=\begin{bmatrix}1 & -2 & -1\\2 & 1 & 2\end{bmatrix}$$

(b) Suppose

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\1\end{bmatrix}+c_3\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Rightarrow\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\1&1&0&0&1&0\\0&1&1&0&0&1\end{array}\right]\begin{array}{l}\\-R1\\ \\ \end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&-1&-1&1&0\\0&1&1&0&0&1\end{array}\right]\begin{array}{l}\\ \\-R2\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&-1&-1&1&0\\0&0&2&1&-1&1\end{array}\right]\begin{array}{l}\\ \\/2\end{array}$$

$$\Rightarrow\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&-1&-1&1&0\\0&0&1&1/2&-1/2&1/2\end{array}\right]\begin{array}{l}-R3\\+R3\\ \\ \end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&1/2&1/2&-1/2\\0&1&0&-1/2&1/2&1/2\\0&0&1&1/2&-1/2&1/2\end{array}\right]$$

$$\Rightarrow\begin{bmatrix}1\\0\\0\end{bmatrix}=1/2\begin{bmatrix}1\\1\\0\end{bmatrix}-1/2\begin{bmatrix}0\\1\\1\end{bmatrix}+1/2\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}=1/2\begin{bmatrix}1\\1\\0\end{bmatrix}+1/2\begin{bmatrix}0\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}=-1/2\begin{bmatrix}1\\1\\0\end{bmatrix}+1/2\begin{bmatrix}0\\1\\1\end{bmatrix}+1/2\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=1/2\begin{bmatrix}1\\2\end{bmatrix}-1/2\begin{bmatrix}-2\\1\end{bmatrix}+1/2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\3/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=1/2\begin{bmatrix}1\\2\end{bmatrix}+1/2\begin{bmatrix}-2\\1\end{bmatrix}-1/2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\1/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=-1/2\begin{bmatrix}1\\2\end{bmatrix}+1/2\begin{bmatrix}-2\\1\end{bmatrix}+1/2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}-2\\1/2\end{bmatrix}$$

$$\Rightarrow[T(\beta, S_2)]=\begin{bmatrix}1 & 0 & -2\\3/2 & 1/2 & 1/2\end{bmatrix}$$

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#2

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \quad S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad T\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x-2y \\ 3x+y \end{bmatrix}$$

Solution:

(a) $I\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad I\left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\Rightarrow [I(\beta, S)] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = P$$

(b) $I\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$I\left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\Rightarrow [I(S, \beta)] = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = Q$$

(c) $T\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T\left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\Rightarrow [T(S, S)] = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = A$$

(d) $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cc:cc} 1 & -1 & 1 & -2 \\ 1 & 1 & 3 & 1 \end{array}\right] \begin{array}{l} \\ -R1 \end{array} \Rightarrow \left[\begin{array}{cc:cc} 1 & -1 & 1 & -2 \\ 0 & 2 & 2 & 3 \end{array}\right] \begin{array}{l} \\ /2 \end{array} \Rightarrow \left[\begin{array}{cc:cc} 1 & -1 & 1 & -2 \\ 0 & 1 & 1 & 3/2 \end{array}\right] \begin{array}{l} +R2 \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 2 & -1/2 \\ 0 & 1 & 1 & 3/2 \end{array}\right]$$

$$\Rightarrow [T(S, \beta)] = \begin{bmatrix} 2 & -1/2 \\ 1 & 3/2 \end{bmatrix} = B.$$

(e) Since $[T(S,\beta)] = [I(S,\beta)][T(S,S)]$ it follows that

$$B = Q \cdot A$$

#3

Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(a+bx+cx^2) = (a+c)+bx^2$. Let $S = \{1, x, x^2\}$ be the standard basis of $P_2(\mathbb{R})$ and $\beta = \{1+x, x, 1+x^2\}$ be another basis.

(a) Find $[T(S,S)]$.

(b) Find $[T(\beta,\beta)]$.

(c) Find an invertible matrix $P$ so that $P[T(\beta,\beta)]P^{-1} = [T(S,S)]$.

Solution:

(a) $T(1) = 1$, $T(x) = x^2$, $T(x^2) = 1$

$$\Rightarrow [T(S,S)] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(b). $T(1+x) = 1+x^2$, $T(x) = x^2$, $T(1+x^2) = 2$. Since

$$1+x^2 = 0\cdot(1+x) + 0\cdot x^2 + 1\cdot(1+x^2)$$
$$x^2 = -1\cdot(1+x) + 1\cdot x + 1(1+x^2)$$
$$2 = 2\cdot(1+x) - 2\cdot x + 0\cdot(1+x^2)$$

it follows that

$$[T(\beta,\beta)] = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

(c). Since $[I(\beta,S)][T(\beta,\beta)][I(S,\beta)] = [T(S,S)]$ it follows that $P = [I(\beta,S)]$ Now,

$$I(1+x) = 1+x,\ I(x) = x\ \ I(1+x^2) = 1+x^2$$

$$\Rightarrow P = [I(\beta,S)] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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#4.

Suppose $\vec{v} \in V$ is a vector in some vector space $V$ and $T: V \to V$ a linear transformation such that $\beta = \{\vec{v}, T(\vec{v}), T^2(\vec{v}), \ldots, T^{n-1}(\vec{v})\}$ is a basis for $V$.

(a) Show that there exists constants $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ such that

$$T^n(\vec{v}) = a_0 \vec{v} + \ldots + a_{n-1} T^{n-1}(\vec{v})$$

(b) Find the matrix $[T(\beta, \beta)]$.

(c) When does $T$ map onto $V$ and when is $T$ one-to-one.

(d) Find the characteristic polynomial $C_T(x)$ of $T$.

Solution:

(a) Since $T^{(n)}(\vec{v}) \in V$ and $\beta$ is a basis there exists $a_0, \ldots, a_{n-1}$ such that

$$T^{(n)}(\vec{v}) = a_0 \vec{v} + \ldots + a_{n-1} T^{n-1}(\vec{v}).$$

(b). Since $T(\vec{v}) = T(\vec{v}),\ T(T(\vec{v})) = T^2(\vec{v}), \ldots, T(T^{n-1}(\vec{v})) = T^n(\vec{v})$ it follows that

$$[T(\beta,\beta)] = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{bmatrix}$$

(c) Since $\det(T(\beta,\beta)) = (-1)^{n+1} a_0$ it follows that $[T(\beta,\beta)]$ is one-to-one and onto if and only if $a_0 \neq 0$.

(d) For $n=2$ we have

$$[T(\beta,\beta)] = \begin{bmatrix} 0 & a_0 \\ 1 & a_1 \end{bmatrix} \Rightarrow \det(\lambda I - [T(\beta,\beta)]) = \det\left(\begin{bmatrix} \lambda & -a_0 \\ -1 & \lambda - a_1 \end{bmatrix}\right) = \lambda(\lambda - a_1) - a_0 = \lambda^2 - a_1\lambda - a_0$$

For $n=3$ we have

$$[T(\beta,\beta)] = \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix} \Rightarrow \det(\lambda I - [T(\beta,\beta)]) = \det\left(\begin{bmatrix} \lambda & 0 & -a_0 \\ -1 & \lambda & -a_1 \\ 0 & -1 & \lambda - a_2 \end{bmatrix}\right)$$

$$= \lambda(\lambda(\lambda - a_2) - a_1) - a_0$$

$$= \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$$

#6.

Prove that 0 is an eigenvalue of $T$ if and only if $\ker(T) \neq \{0\}$.

Solution:

($\Rightarrow$) If 0 is an eigenvalue of $T$ then there exists $\vec{v} \in V$ such that $T\vec{v} = 0 \cdot \vec{v} \Rightarrow \vec{v} \in \ker(T)$.

($\Leftarrow$) If $\vec{v} \in \ker(T)$ then $T\vec{v} = 0 = 0 \cdot \vec{v}$ and thus 0 is an eigenvalue of $T$.

$\blacksquare$

#8

A matrix $N$ is called nilpotent if $N^k = 0$ for some positive integer $k$. Prove that the only possible eigenvalue of a nilpotent matrix is 0.

Solution:

Let $\lambda$ be an eigenvalue of $N$ with eigenvector $\vec{v}$. Therefore,

$$N\vec{v} = \lambda\vec{v}$$

$$\Rightarrow N^k\vec{v} = \lambda^k\vec{v} = 0 = 0 \cdot \vec{v}$$

$$\Rightarrow \lambda = 0.$$

$\blacksquare$