MTH 225: Homework #4

Due Date: February 23, 2024

1. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(a_1, a_2, a_3) = (4a_1 + a_3, 2a_1 + 3a_2 + 2a_3, a_1 + 4a_3).$$

   (a) Let $\mathcal{S} = \{(1,0,0), (0,1,0), (0,.0,1)\}$ be the standard basis of $\mathbb{R}^3$. Find $[T(\mathcal{S}, \mathcal{S})]$ the matrix of T with respect to $\mathcal{S}$.

   (b) Find the characteristic polynomial of T. Are all the roots in $\mathbb{R}$?

   (c) Find the eigenvalues of T.

   (d) Find a basis for the eigenspace of T corresponding to each eigenvalue. What are their dimensions?

   (e) Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ that consists of eigenvectors of T.

   (f) Find $[T(\mathcal{B}, \mathcal{B})]$, the matrix of T with respect to $\mathcal{B}$.

   (g) Find the matrix P so that $[T(\mathcal{B}, \mathcal{B})] = P^{-1}[T(\mathcal{S}, \mathcal{S})]P$

2. If $B = PAP^{-1}$, then prove $B^n = PA^nP^{-1}$ for any $n \in \mathbb{Z}$.

3. Suppose that A and B are $n \times n$ diagonalizable matrices with the same eigenspaces (but not necessarily the same eigenvalues). Prove that $AB = BA$.

4. Let $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be a set of *distinct* eigenvalues of T, and let $\{v_1, v_2, \dots, v_k\}$ be a set of vectors such that v_i is an eigenvector corresponding to the eigenvalue λ_i. Prove that $\{v_1, v_2, \dots, v_k\}$ is a set of linearly independent vectors.

5. Consider $e^x, e^{2x}, \dots, e^{nx}$. Show that each of these functions in $\mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$ is an eigenvector for the differentiation operator. Here, $\mathbb{C}(\mathbb{R}, \mathbb{R})$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

6. Let $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $\vec{v} = (v_1, \dots v_n) \in \mathbb{R}^n$. Let $A = \vec{u}\vec{v}^T$

   (a) Find the columns of A in terms of $\vec{u}$ and $\vec{v}$.

   (b) Show that A is a rank 1 matrix.

7. Give an example of a matrix $A \in M_{4\times 4}(\mathbb{R})$ such that $\text{im}(A) = \ker(A)$. Show that there does not exist a matrix $A \in M_{5\times 5}(\mathbb{R})$ such that $\text{im}(A) = \ker(A)$.

8. Let $\vec{u}, \vec{v} \in \mathbb{C}^n$. **Hint: For this problem you might have to review the geometric interpretation of how vectors are added and subtracted.**

   (a) Prove that $\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

   (b) Prove that if $\vec{u}$ and $\vec{v}$ have the same norm, then $\vec{u} + \vec{v}$ is orthogonal to $\vec{u} - \vec{v}$.

   (c) Prove that the diagonals of a rhombus are orthogonal to each other.

   (d) Prove the following

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).$$

   (e) Prove that the sum of the squares of the length of the four sides of a parallelogram is equal to the sum of the squares of the length of the two diagonals.

9. If $\vec{v}_1 \dots \vec{v}_n$ are mutually orthogonal nonzero vectors, prove that they must be linearly independent.

Homework #4

#1

Consider the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(a_1, a_2, a_3) = (4a_1 + a_3,\ 2a_1 + 3a_2 + 2a_3,\ a_1 + 4a_3)$$

(a) Let S be the standard basis for $\mathbb{R}^3$. Find $[T(S,S)]$.

(b) Find the characteristic polynomial of T.

(c) Find the eigenvalues of T.

(d) Find a basis for each eigenspace of T.

(e) Find a basis β for $\mathbb{R}^3$ consisting of eigenvectors of T.

(f) Find $[T(\beta,\beta)]$.

(g) Find P so that $[T(\beta,\beta)] = P^{-1}[T(S,S)]P$.

Solution:

(a) $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}4\\2\\1\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\0\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\4\end{bmatrix}$

$$\Rightarrow [T(S,S)] = \begin{bmatrix}4 & 0 & 1\\ 2 & 3 & 2\\ 1 & 0 & 4\end{bmatrix}$$

(b) $\det(\lambda I - A) = \det\left(\begin{bmatrix}\lambda-4 & 0 & -1\\ -2 & \lambda-3 & -2\\ -1 & 0 & \lambda-4\end{bmatrix}\right)$

$$= (\lambda-4)\det\left(\begin{bmatrix}\lambda-3 & -2\\ 0 & \lambda-4\end{bmatrix}\right) - 1\det\left(\begin{bmatrix}-2 & \lambda-3\\ -1 & 0\end{bmatrix}\right)$$

$$= (\lambda-4)^2(\lambda-3) - (\lambda-3)$$

$$= (\lambda-3)(\lambda^2 - 8\lambda + 15)$$

$$= (\lambda-3)(\lambda-3)(\lambda-5)$$

$$\Rightarrow c_T(\lambda) = (\lambda-3)^2(\lambda-5)$$

(c) $\lambda = 3, 5$.

(d) $\lambda = 3$

$$\lambda I - A = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \ker(\lambda I - A) = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = -a_3\}$

$\Rightarrow$ If $\vec{a} \in \ker(\lambda I - A)$ then

$$\vec{a} = \begin{bmatrix} -a_3 \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow E_3 = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\lambda = 5$

$$\lambda I - A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ +2R1 \\ +R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} /2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \ker(\lambda I - A) = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3, a_2 = a_3\}$.

$\Rightarrow$ If $\vec{a} \in \ker(\lambda I - A)$ then

$$\vec{a} = \begin{bmatrix} a_3 \\ a_3 \\ a_3 \end{bmatrix} = a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow E_5 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(e) $\beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(f) $[T(\beta, \beta)] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(g) Since $[I(\beta, S)] = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $[T(\beta,\beta)] = [I(S,\beta)][T(S,S)][I(\beta,S)]$

it follows that $P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

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#3.

Suppose that A and B are $n \times n$ diagonalizable matrices with the same eigenspaces. Prove that $AB = BA$.

Solution:

Let A, B be diagonalizable matrices with the same eigenspaces. Therefore there exists diagonal matrices Δ_1, Δ_2 and an invertible matrix P such that

$$A = P^{-1} \Delta_1 P, \quad B = P^{-1} \Delta_2 P.$$

Therefore,

$$\begin{aligned} AB &= P^{-1} \Delta_1 P P^{-1} \Delta_2 P \\ &= P^{-1} \Delta_1 \Delta_2 P \\ &= P^{-1} \Delta_2 \Delta_1 P \\ &= P^{-1} \Delta_2 P P^{-1} \Delta_1 P \\ &= BA. \end{aligned}$$

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#6.

Let $\vec{u} \in \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$, $A = \vec{u}\vec{v}^T$.

(a) Find the columns of A in terms of $\vec{u}$ and $\vec{v}$.

(b) Show that A is a rank 1 matrix.

Solution:

(a) Letting $\{\vec{e}_1, \ldots, \vec{e}_n\}$ denote the standard basis it follows that

$$A\vec{e}_i = \vec{u}\vec{v}^T\vec{e}_i = \vec{u}v_i = v_i\vec{u}$$

$$\Rightarrow A = [v_1\vec{u} \mid v_2\vec{u} \mid \cdots \mid v_n\vec{u}]$$

(b) Since $\operatorname{im}(A) = \operatorname{span}\{\vec{u}\}$ it follows that A is rank 1.

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#7

Give an example of a matrix $A \in M_{4\times 4}(\mathbb{R})$ such that $\mathrm{im}(A) = \mathrm{Ker}(A)$.

Show that there does not exist a matrix $A \in M_{5\times 5}$ such that $\mathrm{im}(A) = \mathrm{Ker}(A)$.

Solution:

Suppose $\mathrm{im}(A) = \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$ and $\mathrm{Ker}(A) = \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$

Therefore, $A\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = A\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, $A\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \in \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$, $A\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \in \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$

Consequently,

$$A = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

works.

$\blacksquare$

#8.

Let $\vec{u}, \vec{v} \in \mathbb{C}^n$.

(a) Prove that $\langle \vec{u}+\vec{v}, \vec{u}-\vec{v} \rangle = \|\vec{u}\|^2 - \|\vec{v}\|^2$

(b) Prove that if $\vec{u}, \vec{v}$ have the same norm, then $\vec{u}+\vec{v}$ is orthogonal to $\vec{u}-\vec{v}$.

(c) Prove that the diagonals of a rhombus are orthogonal to each other.

(d) Prove the following

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$$

(e) Prove that the sum of the squares of the length of the four sides of a parallelogram is equal to the sum of the squares of the length of the two diagonals.

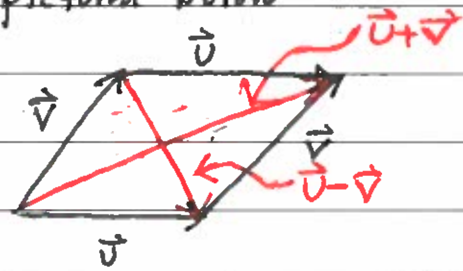
Solution:

(a) $\langle \vec{u}+\vec{v}, \vec{u}-\vec{v} \rangle = \langle \vec{u}, \vec{u}-\vec{v} \rangle + \langle \vec{v}, \vec{u}-\vec{v} \rangle$

$$= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, -\vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle$$
$$= \|\vec{u}\|^2 - \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle - \|\vec{v}\|^2$$
$$= \|\vec{u}\|^2 - \|\vec{v}\|^2.$$

(b) If $\|\vec{u}\| = \|\vec{v}\|$ then $\langle \vec{u}+\vec{v}, \vec{u}-\vec{v} \rangle = 0$.

(c) In $\mathbb{R}^2$ $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ form the diagonals of a parallelogram. as pictured below



In a rhombus $\|\vec{u}\| = \|\vec{v}\|$ and thus $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal

(d) $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle + \langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle$

$$= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle$$
$$+ \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, -\vec{v} \rangle + \langle -\vec{v}, \vec{u} \rangle + \langle -\vec{v}, -\vec{v} \rangle$$
$$= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2$$
$$= 2(\|\vec{u}\|^2 + \|\vec{v}\|^2).$$

(e) The result follows from the same picture in part (c). ∎