MTH 225: Homework #5

Due Date: March 01, 2024

1. Suppose $A, B \in \mathbb{M}_{n\times n}(\mathbb{C})$ are two similar matrices. Prove that A and B have the same characteristic polynomial. **Hint:** For this problem you might have to review properties of determinants.

2. Prove that if $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable then A has a square root, i.e., there exists $B \in M_{n\times n}(\mathbb{C})$ such that $B^2 = A$.

3. Let A and $\vec{v}$ be given by
$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$
   (a) Find an orthonormal basis for $\text{im}(A)$.
   (b) Find an orthonormal basis for $\text{im}(A)^{\perp}$.
   (c) Compute the orthogonal projection of $\vec{v}$ onto $\text{im}(A)$.
   (d) Compute the orthogonal projection of $\vec{v}$ onto $\text{im}(A)^{\perp}$.

4. Let W be a subspace of $\mathbb{C}^n$ and $\{\vec{u}_1, \dots \vec{u}_k\}$ be an orthonormal basis of W.
   (a) Show that for all $\vec{v} \in \mathbb{C}^n$ there exists $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^{\perp}$ such that $\vec{v} = \vec{w}_1 + \vec{w}_2$.
   (b) Prove that $W \cap W^{\perp} = \{0\}$.
   (c) Prove that $\dim(W) + \dim(W^{\perp}) = n$.

5. Prove that if S is a subset of $\mathbb{C}^n$ then $(S^{\perp})^{\perp} = \text{span}(S)$

6. Let $A \in M_{m\times n}(\mathbb{C})$.
   (a) Prove that $\ker(A) \subseteq \ker(A^*A)$.
   (b) Prove that $\ker(A^*A) \subseteq \ker(A)$. **Hint:** If $\vec{v} \in \ker(A^*A)$, what is $\langle A\vec{v}, A\vec{v}\rangle$?
   (c) Prove that $\text{rank}(A^*A) = \text{rank}(A)$.
   (d) Prove that the columns of A are linearly independent if and only if A^*A is invertible.

7. Suppose $A \in M_{n\times n}(\mathbb{C})$ and $z, w \in \mathbb{C}$.
   (a) Prove that $\overline{zw} = \bar{z}\bar{w}$.
   (b) Prove that $\det(A) = \overline{\det(A^*)}$.
   (c) Prove that $|\det(A)|$ equals the product of its singular values.

8. Prove that if λ is an eigenvalue of a unitary matrix then $|\lambda| = 1$.

9. Prove that if $A \in M_{m\times n}(\mathbb{C})$ is a rank 1 matrix then it is of the form $\vec{u}\vec{v}^*$ for some vectors $\vec{u}$ and $\vec{v}$.

10. Determine the singular value decompositions of the following matrices.

(a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $C = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(e) $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

11. If P is a unitary matrix, show that PA has the same singular values as A.

Homework #5

#1

Suppose $A, B \in M_{n\times n}(\mathbb{C})$ are two similar matrices. Prove that A and B have the same characteristic polynomials.

proof:

Suppose A, B are similar matrices. Therefore, there exists $P \in M_{n\times n}(\mathbb{C})$ such that $A = P^{-1}BP$. Consequently,

$$\begin{aligned}\det(\lambda I - A) &= \det(\lambda I - P^{-1}BP)\\ &= \det(\lambda P^{-1}P - P^{-1}BP)\\ &= \det(P^{-1}(\lambda I - B)P)\\ &= \det(P^{-1})\det(\lambda I - B)\det(P)\\ &= \det(P)/\det(P)\det(\lambda I - B)\\ &= \det(\lambda I - B).\end{aligned}$$

$\blacksquare$

#2

Prove that if A is diagonalizable then A has a square root, i.e., there exists $B \in M_{n\times n}(\mathbb{C})$ such that $B^2 = A$.

proof:

Suppose $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable. Therefore, there exists a diagonal matrix $\Delta \in M_{n\times n}(\mathbb{C})$ and $P \in M_{n\times n}(\mathbb{C})$ such that

$$A = P\Delta P^{-1}$$

If we let $\sqrt{\Delta}$ denote the diagonal matrix whose entries are given by $\sqrt{\Delta_{ij}}$ it follows that $B = P\sqrt{\Delta}P^{-1}$ satisfies

$$\begin{aligned}B^2 &= P\sqrt{\Delta}P^{-1}P\sqrt{\Delta}P^{-1}\\ &= P\sqrt{\Delta}\sqrt{\Delta}P^{-1}\\ &= P\Delta P^{-1} = A.\end{aligned}$$

$\blacksquare$

#3.

Let A and $\vec{v}$ be given by

$$A=\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}, \quad \vec{v}=\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

(a) Find an orthonormal basis for $\text{im}(A)$

(b) Find an orthonormal basis for $\text{im}(A)^{\perp}$

(c) Compute the orthogonal projection of $\vec{v}$ onto $\text{im}(A)$.

(d) Compute the orthogonal projection of $\vec{v}$ onto $\text{im}(A)^{\perp}$.

Solution:

(a) Since

$$\text{im}(A)=\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix}\right\}=\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$

we apply G-S to this collection of vectors.

$$\Rightarrow \vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \vec{w}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1$$

Now,

$$\langle \vec{v}_2, \vec{u}_1 \rangle = \frac{1}{\sqrt{5}}(2-1-4-4+2)$$

$$= \frac{-5}{\sqrt{5}} = -\sqrt{5}.$$

Consequently,

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} + \frac{\sqrt{5}}{\sqrt{5}}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\Rightarrow \vec{u}_2 = \vec{w}_2 / \|\vec{w}_2\|$$

$$\Rightarrow \vec{u}_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Continuing we have that

$$\vec{w}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2.$$

Now,

$$\langle \vec{v}_3, \vec{u}_1 \rangle = \frac{1}{\sqrt{5}}(5+4+3+7+1), \quad \langle \vec{v}_3, \vec{u}_2 \rangle = \frac{1}{2}(5-3-7+1)$$

$$= \frac{20}{\sqrt{5}} \qquad\qquad = -2$$

Therefore,

$$\vec{w}_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$\Rightarrow \vec{u}_3 = \vec{w}_3 / \|\vec{w}_3\|$$

$$\Rightarrow \vec{u}_3 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\Rightarrow \operatorname{im}(A) = \operatorname{span}\left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right)$$

$$= \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right)$$

) Consequently, if

$$\vec{w} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \operatorname{im}(A)^\perp$$

then

$$\begin{aligned} x_1 - x_2 - x_3 + x_4 + x_5 &= 0 \\ x_1 + x_3 - x_4 + x_5 &= 0 \\ x_1 + x_3 + x_4 - x_5 &= 0 \end{aligned}$$

$$\Rightarrow \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ -R1 \\ -R1 \end{array} \Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ -R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ /2 \end{array} \Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} +R3 \\ + \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \\ /2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} +R2 \\ \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_4 &= x_5 \\ x_2 &= 2x_4 - 2x_3 = 2x_5 - 2x_3 \\ x_1 &= x_2 = 2x_5 - 2x_3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_5 - 2x_3 \\ 2x_5 - 2x_3 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \operatorname{im}(A)^\perp = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Since these vectors are already orthogonal it follows that an orthonormal basis for $\operatorname{im}(A)^\perp$ is given by

$$\operatorname{im}(A)^{\perp} = \operatorname{span}\left\{\frac{1}{3}\begin{bmatrix}-2\\-2\\1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}2\\2\\0\\1\\1\end{bmatrix}\right\} = \operatorname{span}\{\vec{v}_4, \vec{v}_5\}.$$

(c) $\operatorname{proj}_{\operatorname{im}(A)}(\vec{v}) = \langle\vec{v},\vec{v}_1\rangle\vec{v}_1 + \langle\vec{v},\vec{v}_2\rangle\vec{v}_2 + \langle\vec{v},\vec{v}_3\rangle\vec{v}_3.$

Since,

$$\langle\vec{v},\vec{v}_1\rangle = \tfrac{1}{\sqrt{5}}(1-2-3+4+5)$$
$$= \tfrac{6}{\sqrt{5}}$$
$$\langle\vec{v},\vec{v}_2\rangle = (\tfrac{1}{2}+\tfrac{3}{2}-\tfrac{4}{2}+\tfrac{5}{2})$$
$$= \tfrac{5}{2}$$
$$\langle\vec{v},\vec{v}_3\rangle = (\tfrac{1}{2}+\tfrac{3}{2}+\tfrac{4}{2}-\tfrac{5}{2})$$
$$= \tfrac{3}{2}$$

It follows that

$$\operatorname{proj}_{\operatorname{im}(A)} = \frac{6}{5}\begin{bmatrix}1\\-1\\-1\\1\\1\end{bmatrix} + \frac{5}{4}\begin{bmatrix}1\\0\\1\\-1\\1\end{bmatrix} + \frac{3}{4}\begin{bmatrix}1\\0\\1\\1\\-1\end{bmatrix}$$
$$= \frac{1}{20}\left(\begin{bmatrix}24\\-24\\-24\\24\\24\end{bmatrix} + \begin{bmatrix}25\\0\\25\\-25\\25\end{bmatrix} + \begin{bmatrix}15\\0\\15\\15\\-15\end{bmatrix}\right)$$
$$= \frac{1}{20}\left(\begin{bmatrix}64\\-24\\16\\14\\34\end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix}32\\-12\\8\\7\\17\end{bmatrix}$$

(d) $\operatorname{proj}_{\operatorname{im}(A)^{\perp}}(\vec{v}) = \langle\vec{v},\vec{v}_4\rangle\vec{v}_4 + \langle\vec{v},\vec{v}_5\rangle\vec{v}_5$

Since,

$$\langle\vec{v},\vec{v}_4\rangle = \tfrac{1}{3}(-2-4+3) = -1$$
$$\langle\vec{v},\vec{v}_5\rangle = \tfrac{1}{\sqrt{10}}(2+4+4+5) = \tfrac{15}{\sqrt{10}}$$

It follows that

$$\text{proj}_{\text{im}(A)}(\vec{v}) = -\frac{1}{3}\begin{bmatrix}-2\\-2\\1\\0\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}2\\2\\0\\1\\1\end{bmatrix}$$

$$= \frac{1}{6}\left(\begin{bmatrix}4\\4\\-1\\0\\0\end{bmatrix} + \begin{bmatrix}18\\18\\0\\9\\9\end{bmatrix}\right)$$

$$= \frac{1}{6}\begin{bmatrix}22\\22\\-1\\9\\9\end{bmatrix}$$

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4

Let W be a subspace of $\mathbb{C}^n$ and $\{\vec{v}_1, \dots, \vec{v}_k\}$ an orthonormal basis of W.

(a) Show that for all $\vec{v} \in \mathbb{C}^n$ there exists $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$ such that $\vec{v} = \vec{w}_1 + \vec{w}_2$.

(b) Prove that $W \cap W^\perp = \{0\}$

(c) Prove that $\dim(W) + \dim(W^\perp) = n$.

Solution:

(a) In class we showed that

$$\vec{r} = \vec{v} - \langle \vec{v}, \vec{v}_1 \rangle \vec{v}_1 - \dots - \langle \vec{v}, \vec{v}_k \rangle \vec{v}_k \in S^\perp$$

Therefore, if $\vec{w}_1 = \langle \vec{v}, \vec{v}_1 \rangle \vec{v}_1 + \dots + \langle \vec{v}, \vec{v}_k \rangle \vec{v}_k$ and $\vec{w}_2 = \vec{v} - \langle \vec{v}, \vec{v}_1 \rangle \vec{v}_1 - \dots - \langle \vec{v}, \vec{v}_k \rangle \vec{v}_k$ it follows that $\vec{v} = \vec{w}_1 + \vec{w}_2$ and $\vec{w}_1 \in S$, $\vec{w}_2 \in S^\perp$.

(b). $\vec{v} \in W \cap W^\perp \Leftrightarrow \langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = 0$.

(c). Since for all $\vec{v} \in V$ we have that there exists $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$ such that $\vec{v} = \vec{w}_1 + \vec{w}_2$ it follows that $V = W_1 + W_2$. Since $W_1 \cap W_2 = \{0\}$ it follows $\dim(V) = \dim(W_1) + \dim(W_2)$.

■

#5

Prove that if S is a subset of $\mathbb{C}^n$ then $(S^\perp)^\perp = \text{span}\{S\}$.

proof:

1. Suppose $\vec{v} \in \text{span}\{S\}$. Therefore, there exists $\vec{v}_1, \ldots, \vec{v}_n \in S$ and scalars $c_1, \ldots, c_n$ such that $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Consequently for all $\vec{w} \in S^\perp$ it follows that

$$\begin{aligned} \langle \vec{w}, \vec{v} \rangle &= \langle \vec{w}, c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \rangle \\ &= c_1^* \langle \vec{w}, \vec{v}_1 \rangle + \cdots + c_n^* \langle \vec{w}, \vec{v}_n \rangle \\ &= 0 \end{aligned}$$

Consequently, $\vec{v} \in (S^\perp)^\perp$.

2. Now, suppose $\vec{v} \in (S^\perp)^\perp$. Therefore, for all $\vec{w} \in S^\perp$ it follows that $\langle \vec{v}, \vec{w} \rangle = 0$. By problem #4, $\mathbb{C}^n = S^\perp + \text{span}(S)$ and $\text{span}(S) \cap S^\perp = \{0\}$ therefore if $\vec{v}_1, \ldots, \vec{v}_k$ and $\vec{V}_1, \ldots, \vec{V}_\ell$ are orthonormal bases for $\text{span}(S)$, and $S^\perp$ respectively. Therefore, there exists constants $c_1, \ldots, c_k$ and $b_1, \ldots, b_\ell$ such that

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + b_1\vec{V}_1 + \cdots + b_\ell\vec{V}_\ell$$

$\Rightarrow \langle \vec{v}, \vec{V}_i \rangle = 0 = b_i$

$\Rightarrow \vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$

$\Rightarrow \vec{v} \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. ∎

E6.

Let $A \in M_{m\times n}(\mathbb{C})$.

(a) Prove that $\ker(A) \subseteq \ker(A^*A)$

(b) Prove that $\ker(A^*A) \subseteq \ker(A)$

(c) Prove that $\operatorname{rank}(A^*A) = \operatorname{rank}(A)$

(d) Prove that the columns of A are linearly independent if and only if A^*A is invertible.

Solution:

(a) If $\vec{v} \in \ker(A)$ then $A\vec{v} = 0 \Rightarrow A^*A\vec{v} = A^*0 = 0 \Rightarrow \vec{v} \in \ker(A^*A)$.
Therefore, $\ker(A) \subseteq \ker(A^*A)$.

(b) If $\vec{v} \in \ker(A^*A)$ then $A^*A\vec{v} = 0$. Consequently,

$$\begin{aligned}\langle A\vec{v}, A\vec{v}\rangle &= \vec{v}^*A^*A\vec{v} \\ &= \vec{v}^*0 \\ &= 0.\end{aligned}$$

Therefore, $A\vec{v} = 0$ and thus $\vec{v} \in \ker(A)$. Consequently, $\ker(A^*A) \subseteq \ker(A)$

c) $$\begin{aligned}\operatorname{rank}(A) &= m - \dim(\ker(A)) \\ &= m - \dim(\ker(A^*A)) \\ &= \operatorname{rank}(A^*A).\end{aligned}$$

(d) The columns of A are linearly independent $\Leftrightarrow \ker(A) = \{0\}$
$\Leftrightarrow \ker(A^*A) = \{0\}$
$\Leftrightarrow A^*A$ is invertible.

■

#7

Suppose $A \in M_{n\times n}(\mathbb{C})$ and $z, w \in \mathbb{C}$.

(a) Prove that $\overline{zw} = \bar{z}\bar{w}$.

(b) Prove that $\det(A) = \overline{\det(A^*)}$

(c) Prove that $|\det(A)| =$ the product of its singular values.

proof

(a) Let $z = a+ib$, $w = c+id$. Therefore,

$$\begin{aligned}\overline{zw} &= \overline{(a+ib)(c+id)} \\ &= \overline{(ac+ibc+ida-bd)} \\ &= (ac-ibc-ida-bd) \\ &= (a-ib)(c-id) \\ &= \bar{z}\cdot\bar{w}\end{aligned}$$

(b) $$\begin{aligned}\det(A^*) &= \det(\overline{A^T}) \\ &= \overline{\det(A^T)} \\ &= \overline{\det(A)}\end{aligned}$$

$$\Rightarrow \overline{\det(A^*)} = \det(A).$$

(c). The singular value decomposition of A is

$$A = U\begin{pmatrix}\sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n\end{pmatrix}V^*$$

$$\begin{aligned}\Rightarrow |\det(A)| &= |\det(U)|\cdot\left|\det\begin{pmatrix}\sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n\end{pmatrix}\right| |\det(V^*)| \\ &= \sigma_1\cdot\sigma_2\cdots\sigma_n.\end{aligned}$$

∎

E8.

Prove that if λ is an eigenvalue of a unitary matrix then $|\lambda|=1$.

proof:

Let U be a unitary matrix with eigenvalue λ and corresponding eigenvector $\vec{v}$. Therefore,

$$U\vec{v}=\lambda\vec{v}$$

Now, since U is unitary it follows that

$$\langle U\vec{v}, U\vec{v}\rangle = \langle \vec{v},\vec{v}\rangle = \|\vec{v}\|^2$$
$$\Rightarrow \langle \lambda\vec{v}, \lambda\vec{v}\rangle = \|\vec{v}\|^2$$
$$\Rightarrow \lambda\cdot\bar{\lambda}\langle \vec{v},\vec{v}\rangle = \|\vec{v}\|^2$$
$$\Rightarrow |\lambda|^2\|\vec{v}\|^2 = \|\vec{v}\|^2$$
$$\Rightarrow |\lambda|^2 = 1$$
$$\Rightarrow |\lambda| = 1.$$

E9.

Prove that if $A\in M_{mn}(\mathbb{C})$ is rank 1 then it is of the form $\vec{U}\vec{V}^*$ for some vectors $\vec{U}$ and $\vec{v}$.

proof:

If A is rank 1 then there exists $\vec{V}\in\mathbb{C}^n$ such that $\mathrm{im}(A)=\mathrm{span}\{\vec{V}\}$.

Consequently, there exists scalars $\vec{U}_1,\dots,\vec{U}_n\in\mathbb{C}^n$ such that

$$A\cdot\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}=U_1^*\vec{V},\quad A\cdot\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}=U_2^*\vec{V},\ \dots,\ A\cdot\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}=U_n^*\vec{V}$$

Consequently,

$$A=\begin{bmatrix}U_1^*\vec{V}_1 & \cdots & U_n^*\vec{V}_n\end{bmatrix}$$
$$=\vec{U}^*\vec{V}$$

Where $\vec{U}=\begin{bmatrix}U_1\\\vdots\\U_n\end{bmatrix}$. ■

#10

Determine the singular value decompositions of the following matrices

(a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $C = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

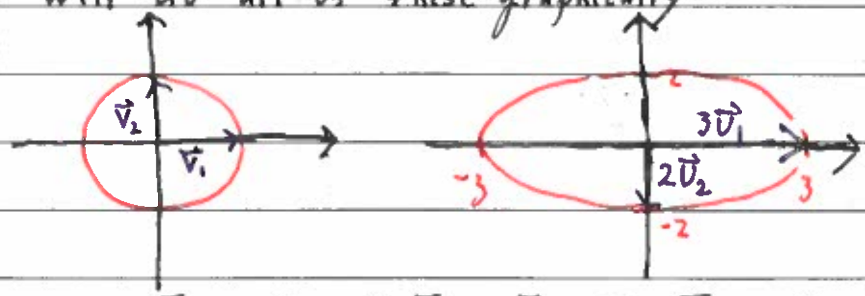
(d) $D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(e) $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Solution:

I will do all of these graphically

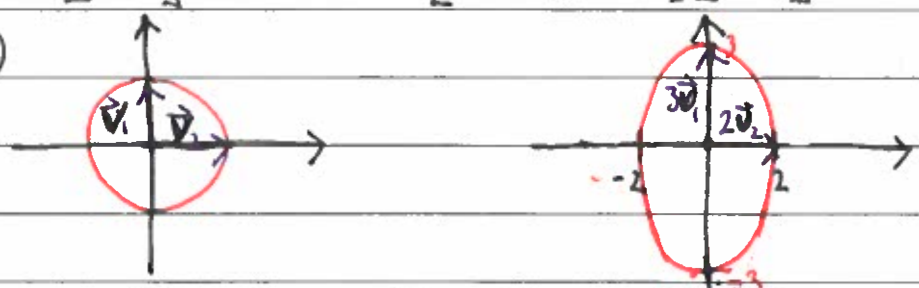
(a)



$$\Rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

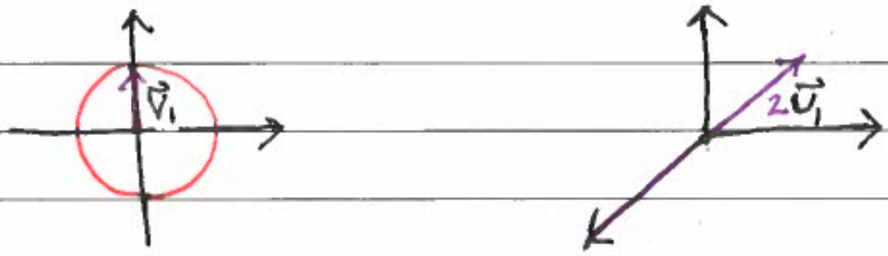
$$\Rightarrow \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = U \Sigma V^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)



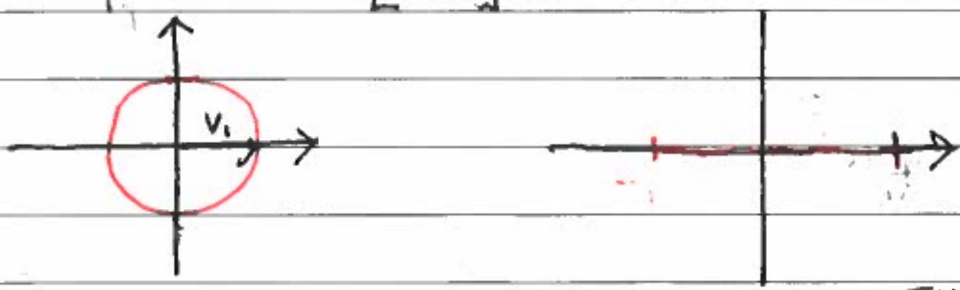
$$\Rightarrow U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c)



$$U=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \Sigma=\begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix},\ V=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

(d)



The maximum direction is achieved when $\vec{v}=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$. That is

$$A\cdot\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}\sqrt{2}\\0\end{bmatrix}$$

$$\Rightarrow U=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \Sigma=\begin{bmatrix}\sqrt{2}&0\\0&0\end{bmatrix},\ V=\begin{bmatrix}1/\sqrt{2}&-1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}$$

#11

If P is a unitary matrix, show that PA has the same singular values as A.

proof:

Let $A=U\Sigma V^*$ be the singular value decomposition of A. Therefore,

$$PA=PU\Sigma V^*$$

Thus, $\tilde{U}\Sigma V^*$ is the singular value decomposition of PA where $\tilde{U}=PU$.