MTH 225: Homework #6

Due Date: March 08, 2024

1. Let $V = C^\infty([-1,1])$, i.e., the vector space of infinitely differentiable real valued functions defined on the interval $[-1,1]$. Define the operation $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R}$ by
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$
and let $U \subset V$ be the subspace defined by $U = \text{span}\{1, x, x^2, x^3\}$.
   (a) Show that the operation $\langle \cdot, \cdot \rangle$ defined above is an inner product on V.
   (b) Using the above inner product, find an orthonormal basis for U.
2. Suppose $A, B \in M_{n\times n}(\mathbb{C})$ are unitary matrices. Prove that AB is a unitary matrix.
3. Suppose that $A = P\Sigma Q^*$ is a singular value decomposition of A. What is a singular value decomposition for A^*?
4. Suppose $A \in M_{n\times n}(\mathbb{C})$.
   (a) Prove that $\ker(A^*) = (\text{im}(A))^\perp$.
   (b) Prove that $\text{im}(A^*) = (\ker(A))^\perp$.
   (c) Prove that $\ker(A) = (\text{im}(A^*))^\perp$.
   (d) Prove that $\text{im}(A) = (\ker(A^*))^\perp$.
   (e) Let $\vec{b} \in \mathbb{C}^n$. Prove that there exists a $\vec{v} \in \mathbb{C}^n$ that satisfies $A\vec{v} = \vec{b}$ if and only if $\langle b, \vec{w} \rangle = 0$ for all $\vec{w}$ satisfying $A^*\vec{w} = 0$.
5. Suppose $A, B \in M_{n\times n}(\mathbb{C})$.
   (a) Prove that $A + A^*$ and $i(A - A^*)$ are Hermitian matrices.
   (b) Prove that A can be written as a linear combination of two Hermitian matrices. These two Hermitian matrices are sometimes called the real and imaginary components of a matrix.
   (c) Prove that if A and B are Hermitian then AB is Hermitian if and only if $AB = BA$, i.e., the matrices commute.
   (d) Prove that if A and B are Hermitian matrices then $AB+BA$ and $i(AB-BA)$ are also Hermitian matrices.
   (e) Prove that if A and B are Hermitian matrices then the matrix C defined by $AB - BA = iC$ is also Hermitian.
6. Suppose $A \in M_{n\times n}(\mathbb{C})$ is both Hermitian and unitary. Prove that its eigenvalues are all ± 1.
7. A matrix $A \in M_{n\times n}(\mathbb{C})$ is called **normal** if $AA^* = A^*A$.
   (a) Prove that Hermitian matrices and unitary matrices are normal matrices.
   (b) Show that the matrix
$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$
is not a symmetric, Hermitian or unitary matrix, but is a normal matrix.
   (c) Show that A above has an orthogonal basis of eigenvectors. Are the eigenvalues real numbers?
   (d) Show that the following matrix is a normal matrix and has an orthogonal basis of eigenvectors
$$B = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Homework #6

#1.

Let $V = C^{\infty}([-1,1])$ and define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ by
$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$$
and let $U = \text{span}\{1, x, x^2, x^3\}$.

(a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.

(b) Find an orthonormal basis for U.

Solution:

(a) Let $f, g, h \in V$ a...

1. $$\begin{aligned} \langle f, g \rangle &= \int_{-1}^{1} f(x) g(x) \, dx \\ &= \int_{-1}^{1} g(x) f(x) \, dx \\ &= \langle g, f \rangle. \end{aligned}$$

2. $$\begin{aligned} \langle f+g, h \rangle &= \int_{-1}^{1} (f(x) + g(x)) h(x) \, dx \\ &= \int_{-1}^{1} (f(x) h(x) + g(x) h(x)) \, dx \\ &= \langle f, h \rangle + \langle g, h \rangle. \end{aligned}$$

3. $$\begin{aligned} \langle af, g \rangle &= \int_{-1}^{1} a f(x) g(x) \, dx \\ &= a \int_{-1}^{1} f(x) g(x) \, dx \\ &= a \langle f, g \rangle \end{aligned}$$

4. $\langle f, f \rangle = \int_{-1}^{1} f(x)^2 \, dx \geq 0$

   The above is zero if and only if $f(x) = 0$.

By items 1-4, $\langle \cdot, \cdot \rangle$ is an inner product.

(b) Let $\vec{u}_1 = 1$, $\vec{u}_2 = x$, $\vec{u}_3 = x^2$, $\vec{u}_4 = x^3$. Therefore,
$$\|u_1\|^2 = \int_{-1}^{1} 1 \, dx = 2$$
Consequently, we let
$$\vec{v}_1 = 1/\sqrt{2}\, \vec{u}_1 = 1/\sqrt{2}.$$

Let

$$\vec{w}_2 = \vec{u}_2 - \langle \vec{u}_2, \vec{v}_1 \rangle \vec{v}_1$$

Since

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \tfrac{1}{\sqrt{2}} x \, dx = 0$$

it follows that

$$\vec{w}_2 = x.$$

Now,

$$\|\vec{w}_2\|^2 = \int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 = 2/3.$$

Therefore,

$$\vec{v}_2 = \sqrt{\tfrac{3}{2}}\, x$$

Let

$$\vec{w}_3 = \vec{u}_3 - \langle \vec{u}_3, \vec{v}_1 \rangle \vec{v}_1 - \langle \vec{u}_3, \vec{v}_2 \rangle \vec{v}_2.$$

Since

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} 1/\sqrt{2}\, x^2 dx = 2\int_0^1 1/\sqrt{2}\, x^2 dx = \sqrt{2}/3$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \sqrt{3/2}\, x^3 dx = 0$$

it follows that

$$\vec{w}_3 = x^2 - 1/3 = 1/3(3x^2 - 1).$$

Now,

$$\|\vec{w}_3\|^2 = 1/9\left(\int_{-1}^{1} 9x^4 - 6x^2 + 1\right)dx$$

$$= 2/9 \int_0^1 (9x^4 - 6x^2 + 1)dx$$

$$= 2/9\,(9/5 - 6/3 + 1)$$

$$= 2/9\,(9/5 - 1)$$

$$= 2/9 \cdot 4/5$$

$$\Rightarrow \|\vec{w}_3\| = 2\sqrt{2}/3\cdot\sqrt{5}$$

Therefore,

$$\vec{v}_3 = \frac{3\sqrt{5}}{2\sqrt{2}}\,\vec{w}_3 = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$$

Let

$$\vec{w}_4 = \vec{u}_4 - \langle \vec{u}_4, \vec{v}_1 \rangle \vec{v}_1 - \langle \vec{u}_4, \vec{v}_2 \rangle \vec{v}_2 - \langle \vec{u}_4, \vec{v}_3 \rangle \vec{v}_3$$

Since

$$\langle \vec{u}_4, \vec{v}_1 \rangle = \int_{-1}^{1} x^3 \cdot 1/\sqrt{2}\, dx = 0$$

$$\langle \vec{u}_4, \vec{v}_2 \rangle = \int_{-1}^{1} x^3 \cdot \sqrt{3/2}\, x\, dx = \sqrt{6} \int_0^1 x^4 dx = \sqrt{6}/5$$

$$\langle \vec{u}_4, \vec{v}_3 \rangle = \int_{-1}^{1} \sqrt{5}/2\sqrt{2}\, (3x^2-1) x^3 dx = 0$$

it follows that

$$\vec{w}_4 = x^3 - \sqrt{6}/5 \cdot \sqrt{3/2}\, x$$

$$= x^3 - 3/5\, x$$

$$= \tfrac{1}{5}(5x^3 - 3x)$$

Now,

$$\|\vec{w}_4\|^2 = 1/25 \int_{-1}^{1} (5x^3 - 3x)^2 dx$$

$$= 2/25 \int_0^1 (25x^6 - 30x^4 + 9x^2) dx$$

$$= 2/25 (25/7 - 30/5 + 3)$$

$$= 2/25 \left(125 - 210 + 105/35 \right)$$

$$= 2 \cdot 20/25 \cdot 35$$

$$= 4 \cdot 2/25 \cdot 7$$

$$\Rightarrow \|\vec{w}_4\| = 2\sqrt{2}/5 \cdot \sqrt{7}$$

Therefore,

$$\vec{v}_4 = \frac{5\sqrt{7}}{2\sqrt{2}} \cdot \frac{1}{5} (5x^3 - 3x)$$

$$= \frac{\sqrt{7}}{2\sqrt{2}} (5x^3 - 3x).$$

Consequently, an orthonormal basis for U is given by

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \frac{\sqrt{5}}{2\sqrt{2}}(3x^2-1), \frac{\sqrt{7}}{2\sqrt{2}}(5x^3-3x) \right\}$$

$\blacksquare$

#2.

Suppose $A, B \in M_{n\times n}(\mathbb{C})$ are unitary matrices. Prove that AB is a unitary matrix.

Solution:

Since A, B are unitary it follows that

$$
\begin{aligned}
(AB)^*AB &= B^*A^*AB \\
&= B^*IB \\
&= B^*B \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
(AB)(AB)^* &= AB \cdot B^*A^* \\
&= AIA^* \\
&= AA^* \\
&= I.
\end{aligned}
$$

Therefore, $(AB)^{-1} = (AB)^*$ and thus AB is unitary.

∎

#3.

Suppose that $A = P\Sigma Q^*$ is a singular value decomposition of A. What is the singular value decomposition of A^*?

Solution:

$$
\begin{aligned}
A^* = (P\Sigma Q^*)^* &= Q\Sigma^*P^* \\
&= Q\Sigma P^*
\end{aligned}
$$

#4.

Suppose $A \in M_{n\times n}(\mathbb{C})$.

(a) Prove that $\ker(A^*) = (\mathrm{im}(A))^\perp$.

(b) Prove that $\mathrm{im}(A^*) = (\ker(A))^\perp$

(c) Prove that $\ker(A) = \mathrm{im}(A^*)^\perp$

(d) Prove that $\mathrm{im}(A) = \ker(A^*)^\perp$

(e) Let $\vec{b} \in \mathbb{C}^n$. Prove that there exists $\vec{v} \in \mathbb{C}^n$ that satisfies $A\vec{v} = \vec{b}$ if and only if $\langle \vec{b}, \vec{w} \rangle = 0$ for all $\vec{w}$ satisfying $A^*\vec{w} = 0$.

Solution:

(a) $\vec{u} \in \ker(A^*)$ if and only if $A^*\vec{u} = 0$ and $\vec{v} \in \mathrm{im}(A)$ if and only if there exists $\vec{x} \in \mathbb{C}^{n\times n}$ such that $A\vec{x} = \vec{v}$. Consequently, $\vec{u} \in \ker(A^*)$ if and only if for all $\vec{v} \in \mathrm{im}(A)$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= \langle \vec{u}, A\vec{x} \rangle \\ &= \langle A^*\vec{u}, \vec{x} \rangle \\ &= \langle 0, \vec{x} \rangle \\ &= 0. \end{aligned}$$

Therefore, $\vec{u} \in \ker(A^*)$ if and only if $\vec{u} \in \mathrm{im}(A)^\perp$.

(b) $\vec{u} \in \mathrm{im}(A^*)$ if and only if there exists $\vec{x} \in \mathbb{C}^n$ such that $A\vec{x} = \vec{u}$ and $\vec{v} \in \ker(A)$ if and only if $A\vec{v} = 0$. Consequently, $\vec{u} \in \mathrm{im}(A^*)$ if and only if for all $\vec{v} \in \ker(A)$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= \langle A^*\vec{x}, \vec{v} \rangle \\ &= \langle \vec{x}, A\vec{v} \rangle \\ &= \langle \vec{x}, 0 \rangle \\ &= 0. \end{aligned}$$

(c) Since $\mathrm{im}(A^*) = \ker(A)^\perp$ it follows that $\mathrm{im}(A^*)^\perp = (\ker(A)^\perp)^\perp = \ker(A)$

(d) Since $\ker(A^*) = \mathrm{im}(A)^\perp$ it follows that $\ker(A^*)^\perp = (\mathrm{im}(A)^\perp)^\perp = \mathrm{im}(A)$.

(e) Follows from part (a).

#5

Suppose $A, B \in M_{n\times n}(\mathbb{C})$.

(a) Prove that $A+A^*$ and $i(A-A^*)$ are Hermitian matrices.

(b) Prove that A can be written as a linear combination of two Hermitian matrices.

(c) Prove that if A, B are Hermitian then AB is Hermitian if and only if $AB=BA$.

(d) Prove that if A and B are Hermitian matrices then $AB+BA$ and $i(AB-BA)$ are also Hermitian.

(e) Prove that if A and B are Hermitian matrices then the matrix C defined by $AB-BA=iC$ is also Hermitian.

Solution:

(a) $(A+A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$

$(i(A-A^*))^* = -i(A^* - (A^*)^*) = -i(A^* - A) = i(A - A^*)$.

(b) $A = \frac{1}{2}(A+A^*) + \frac{1}{2i}\, i(A-A^*)$

(c) $(AB)^* = B^*A^* = BA$ and thus AB is Hermitian if and only if $AB=BA$.

(d) $(AB+BA)^* = B^*A^* + A^*B^* = BA + AB = AB+BA$.

$(i(AB-BA))^* = -i(B^*A^* - A^*B^*) = -i(BA-AB) = i(AB-BA)$.

(e) Since $C = \frac{1}{i}(AB-BA) = -i(AB-BA) = i(BA-AB)$ it follows from part (d) that C is Hermitian. ∎

#6.

Suppose $A \in M_{n\times n}(\mathbb{C})$ is both Hermitian and unitary. Prove that its eigenvalues are all ± 1.

~~proof~~:

Eigenvalues of unitary matrices satisfy $|\lambda|=1$. Since eigenvalues of Hermitian matrices are real it follows that $\lambda = \pm 1$. ∎

#7.

A matrix $A \in M_{n\times n}(\mathbb{C})$ is normal if $AA^* = A^*A$.

(a) Prove that Hermitian matrices and unitary matrices are normal.

(b) Show that $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ is not symmetric, Hermitian, or unitary but is a normal matrix.

(c) Show that A has an orthogonal basis of eigenvectors.

(d) Show that the following matrix is a normal matrix and has an orthogonal basis of eigenvectors

$$B = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

Solution:

(a) If A is Hermitian then

$$AA^* = AA = A^*A$$

If A is unitary then

$$AA^* = I = A^*A.$$

(b) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ is clearly not Hermitian or symmetric. It is not unitary since the columns are not orthonormal. However

$$A^*A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$AA^* = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

(c) Since $Tr(A) = \lambda_1 + \lambda_2 = 0$ and $det(A) = \lambda_1\lambda_2 = 4$ it follows that $\lambda_1 = 2i$, $\lambda_2 = -2i$ are the eigenvalues.

$\underline{\lambda_1 = 2i}$

$$\lambda I - A = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \xrightarrow{\times i} \begin{bmatrix} -2 & 2i \\ -2 & 2i \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\underline{\lambda_1 = -2i}$

$$\lambda I - A = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \xrightarrow{\times i} \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

(d) If $B = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ then $B^* = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ and thus

$$B \cdot B^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B^* B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Therefore B is normal. Since $\mathrm{Tr}(B) = 2$ and $\mathrm{Det}(B) = 2$ it follows that $\lambda_1 + \lambda_2 = 2$ and $\lambda_1 \lambda_2 = 2$.

$$\Rightarrow \lambda_1 + \frac{2}{\lambda_1} = 2$$

$$\Rightarrow \lambda_1^2 - 2\lambda_1 + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow \lambda = 1 \pm i$$

$\underline{\lambda = 1+i}$

$$\lambda I - B = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 1-i}$

$$\lambda I - B = \begin{bmatrix} -i & -i \\ -i & -i \end{bmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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