MTH 225: Homework #8

Due Date: April 12, 2024

1. Let $A \in M_{2\times 2}(\mathbb{C})$ be given by
$$A = \begin{bmatrix} 5/2 & i/2 \\ -i/2 & 5/2 \end{bmatrix}.$$
   (a) Show that A is Hermitian.
   (b) Find the eigenvalues and eigenvectors of A.
   (c) Show that A is positive semidefinite.
   (d) Find the square root of A.

2. Let $A \in M_{2\times 2}(\mathbb{C})$ be given by
$$A = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}.$$
   (a) Find an orthonormal basis for $\ker(A)$.
   (b) Find an orthonormal basis for $\operatorname{im}(A)$.
   (c) Find the SVD of A.
   (d) Find the polar form of A.

3. Find $\exp(A)$ when A is given by
   (a) $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$
   (c) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

   Hint: The third matrix is not diagonalizable. You will have to use the definition of the $\exp(A)$ and find a pattern for the powers of A.

4. Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $A\exp(A) = \exp(A)A$.

5. Prove that if O is the $n \times n$ zero matrix then $\exp(O) = I$.

6. Prove for all $A \in M_{n\times n}(\mathbb{C})$ and $a, b \in \mathbb{R}$ that $\exp((a+b)A) = \exp(aA)\exp(bA)$.

7. Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $\exp(-A) = \exp(A)^{-1}$.

8. Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $\exp(A^*) = \exp(A)^*$.

9. Prove that if $A \in M_{n\times n}(\mathbb{C})$ satisfies $A^* = -A$ then $\exp(A)$ is unitary.

10. Prove that if $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable then
$$\det(\exp(A)) = e^{\mathrm{Tr}(A)}.$$

   Hint: Recall that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A then $\mathrm{Tr}(A) = \lambda_1 + \ldots + \lambda_n$.

11. Consider the inconsistent system $A\vec{x} = \vec{b}$ when

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

(a) Find the associated normal equations.

(b) Find the least square solution to this system.

12. The median price (in thousands of dollars) of existing homes in a certain metropolitan area from 1989 to 1999 is summarized in the following table:

year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999,
price	86.4	89.8	92.8	96.0	99.6	103.1	106.3	109.5	113.3	120.0	129.5 .

(a) Set up the normal equations to find the least squares line for these data. **Hint:** Use a calculator or any kind of software to simplify the matrix computations.

(b) Solve the normal equations to find the least squares line for these data.

(c) Use your least squares line to estimate the median price of a house in 2005 and 2010.

13. A 20 pound turkey that is at the room temperature of 72° is placed in the oven at 1 : 00 PM. The temperature of the turkey is observed in 20 minute intervals to be 79°, 88°, and 96°. A turkey is cooked when its temperature reaches 165°. Using least squares, estimate how much longer you have to wait until the turkey is done cooking.

Homework #8

#1.

Let $A \in M_{2\times 2}(\mathbb{C})$ be given by

$$A = \begin{bmatrix} 5/2 & i/2 \\ -i/2 & 5/2 \end{bmatrix}$$

(a) Show that A is Hermitian

(b) Find the eigenvalues and eigenvectors of A

(c) Show that A is positive definite.

(d) Find the square root of A.

Solution:

(a) $A^* = \begin{bmatrix} 5/2 & i/2 \\ -i/2 & 5/2 \end{bmatrix}^* = \begin{bmatrix} 5/2 & i/2 \\ -i/2 & 5/2 \end{bmatrix}$

(b) $\det(\lambda I - A) = \det\begin{pmatrix} \lambda - 5/2 & -i/2 \\ i/2 & \lambda - 5/2 \end{pmatrix} = \lambda^2 - 5\lambda + 25/4 - 1/4 = \lambda^2 - 5\lambda + 6.$

Therefore, $\lambda_1 = 3$, $\lambda_2 = 2$.

$\lambda_1 = 3$

$\lambda_1 - I = \begin{bmatrix} 3-5/2 & -i/2 \\ i/2 & 3-5/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix}$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Normalizing, we have $\vec{v}_1 = 1/\sqrt{2} \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = 2$

Since eigenvectors of Hermitian matrices are orthonormal:

$\vec{v}_2 = 1/\sqrt{2} \begin{bmatrix} -i \\ 1 \end{bmatrix}$

(c) Since $\lambda_1, \lambda_2 \geq 0$ it follows that A is positive definite.

(d) By parts (a)-(c) we have that

$$A = V \Lambda V^{-1}$$
$$= \begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
$$\Rightarrow \sqrt{A} = \begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -i\sqrt{3}/\sqrt{2} & \sqrt{3}/\sqrt{2} \\ i\sqrt{2}/\sqrt{2} & \sqrt{2}/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{3}/2 + \sqrt{2}/2 & i(\sqrt{3}-\sqrt{2})/2 \\ i(-\sqrt{3}+\sqrt{2})/2 & \sqrt{3}/2 + \sqrt{2}/2 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} \sqrt{3}+\sqrt{2} & i(\sqrt{3}-\sqrt{2}) \\ -i(\sqrt{3}-\sqrt{2}) & \sqrt{3}+\sqrt{2} \end{bmatrix}$$

#2.

Let $A \in M_{2\times 2}(\mathbb{C})$ be given by

$$A = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$$

(a) Find an orthonormal basis for $\ker(A)$

(b) Find an orthonormal basis for $\operatorname{im}(A)$.

(c) Find the SVD of A.

(d) Find the polar form of A.

Solution:

(a) $\ker(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$.

(b) $\operatorname{im}(A) = \operatorname{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}\right\}$.

(c) We know that $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ with singular value $\sigma_2 = 0$. Therefore, $\vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$
$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -6/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix} = \frac{2}{\sqrt{2}} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = 2\sqrt{5} \begin{bmatrix} -3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$$

Consequently,

$$\sigma_1 = 2\sqrt{5}, \quad \vec{u}_1 = \begin{bmatrix} -3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}.$$

We also have from orthonormality that

$$\vec{u}_2 = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

Therefore,

$$A = U \Sigma V^*$$
$$= \begin{bmatrix} -3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(d) Now, $A^*A = V\Sigma^2 V^* \Rightarrow \sqrt{A^*A} = V \Sigma V^*$. Therefore,

$$A = U \Sigma V^*$$
$$= U V^* V \Sigma V^*$$
$$= Q\sqrt{A^*A},$$

where $Q = UV^*$. Computing we have that

$$Q = \begin{bmatrix} -3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 2/\sqrt{20} & -4/\sqrt{20} \\ 4/\sqrt{20} & 2/\sqrt{20} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} & -2/\sqrt{10} \\ 2/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

$$\sqrt{A^*A} = \begin{bmatrix} -3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{2\sqrt{5}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} -3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} -5^{1/4} & 5^{1/4} \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3/\sqrt{2\sqrt{5}} & -3/\sqrt{2\sqrt{5}} \\ 1/\sqrt{2\sqrt{5}} & -1/\sqrt{2\sqrt{5}} \end{bmatrix}$$

$\blacksquare$

#3.

Find $\exp(A)$ when A is given by

(a) $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Solution:

(a) The eigenvalues of A are given by $\lambda_1 = 2$, $\lambda_2 = -1$.

$\lambda_1 = 2$

$$\lambda_1 I - A = \begin{bmatrix} 0 & -1 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda_2 = -1$

$$\lambda_2 I - A = \begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}, \text{ or } \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Therefore,

$$V = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \Rightarrow V^{-1} = \begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \end{bmatrix}$$

Consequently,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \end{bmatrix}$$

$$\Rightarrow \exp(A) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} e^2 & e^2/3 \\ 0 & e^{-1}/3 \end{bmatrix}$$

$$= \begin{bmatrix} e^2 & e^2/3 - e^{-1}/3 \\ 0 & e^{-1} \end{bmatrix}$$

(b). The eigenvalues satisfy

$$\lambda_1+\lambda_2=0, \quad \lambda_1\lambda_2=4$$
$$\Rightarrow -\lambda_1^2=4 \Rightarrow \lambda_1=\pm 2i$$

$\underline{\lambda_1=2i:}$

$$\lambda I-A=\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \Rightarrow \vec{V}_1=\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\underline{\lambda_1=-2i:}$

$$\lambda I-A=\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \Rightarrow \vec{V}_2=\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

$$\Rightarrow V=\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \Rightarrow V^{-1}=\begin{bmatrix} 1/2 & 1/2\, i \\ 1/2 & -1/2\, i \end{bmatrix}$$

Consequently,

$$A=\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}\begin{bmatrix} 1/2 & 1/2\, i \\ 1/2 & -1/2\, i \end{bmatrix}$$

$$\Rightarrow \exp(A)=\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{2i} & 0 \\ 0 & e^{-2i} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2\, i \\ 1/2 & -1/2\, i \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{2i}/2 & i e^{2i}/2 \\ e^{-2i}/2 & -ie^{-2i}/2 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{2}(e^{2i}+e^{-2i}) & i/2(e^{2i}-e^{-2i}) \\ -i/2(e^{2i}-e^{-2i}) & 1/2(e^{2i}+e^{-2i}) \end{bmatrix}$$

$$=\begin{bmatrix} \cos(2) & -\sin(2) \\ \sin(2) & \cos(2) \end{bmatrix}$$

(c) If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

We also have that

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4+2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}.$$

$$A^4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix} =$$

For $k \in \mathbb{N}$, let $P(k)$ be the logical statement that $A^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$.

1. Clearly, $P(1)$ is true.
2. If we assume that $P(k)$ is true it follows that

$$A \cdot A^k = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2k+2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2(k+1) \\ 0 & 1 \end{bmatrix}$$

Consequently, $P(k+1)$ is true

By items 1 and 2 and the Principle of Mathematical Induction it follows that for all $k \in \mathbb{N}$

$$A^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$\exp(A) = I + A + \tfrac{1}{2}A^2 + \tfrac{1}{6}A^3 + \dots + \tfrac{1}{n!}A^n + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} + \dots + \frac{1}{n!}\begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 + \frac{1}{2} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots & 2 + 4/2 + 6/3! + \dots + 2n/n! + \dots \\ 0 & 1 + \frac{1}{2} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots \end{bmatrix}$$

$$= \begin{bmatrix} e^1 & 2(1 + 1 + 3/3! + \dots + 1/(n-1)! + \dots) \\ 0 & e^1 \end{bmatrix}$$

$$= \begin{bmatrix} e^1 & 2(1 + 1 + 1/2 + \dots + 1/(n-1)! + \dots) \\ 0 & e^1 \end{bmatrix}$$

$$= \begin{bmatrix} e^1 & 2e^1 \\ 0 & e^1 \end{bmatrix}$$

#4.

Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $A\exp(A) = \exp(A)A$.

~~proof~~:

$$A\exp(A) = A(I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \dots)$$
$$= A + A^2 + \tfrac{1}{2}A^3 + \tfrac{1}{3!}A^4 + \dots$$
$$= (I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \dots)A$$
$$= \exp(A)A.$$

■

#5.

Prove that if O is the $n\times n$ zero matrix then $\exp(O) = I$.

~~proof~~:

$$\exp(O) = I + O + \tfrac{1}{2}O^2 + \tfrac{1}{3!}O^3 + \dots$$
$$= I + O + O + O + \dots$$
$$= I$$

■

#6.

Prove for all $A \in M_{n\times n}(\mathbb{C})$ and $a, b \in \mathbb{R}$ that

$$\exp((a+b)A) = \exp(aA)\exp(bA).$$

~~proof~~:

Computing, we have that

$$\exp(aA)\exp(bA) = (I + aA + \tfrac{1}{2}a^2A^2 + \tfrac{1}{3!}a^3A^3 + \dots)(I + bA + \tfrac{1}{2}b^2A^2 + \tfrac{1}{3!}b^3A^3 + \dots)$$
$$= I + (a+b)A + \tfrac{1}{2}(b^2 + 2ab + a^2)A^2 + \tfrac{1}{3!}(b^3 + 3ab^2 + 3a^2b + a^3)A^3 + \dots$$
$$= I + (a+b)A + \tfrac{1}{2}(a+b)^2A^2 + \tfrac{1}{3!}(a+b)^3A^3 + \dots$$
$$= \exp((a+b)A).$$

■

#7.

Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $\exp(-A) = \exp(A)^{-1}$.

proof:

$$I = \exp(0) = \exp(A-A) = \exp(A)\exp(-A)$$

$$\Rightarrow \exp(A)^{-1} = \exp(-A).$$

$\blacksquare$

#8.

Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $\exp(A^*) = \exp(A^*)$.

proof:

$$\begin{aligned}\exp(A)^* &= \left(I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \dots\right)^* \\ &= I^* + A^* + \tfrac{1}{2}(A^2)^* + \tfrac{1}{3!}(A^3)^* + \dots \\ &= I + A^* + \tfrac{1}{2}(A^*)^2 + \tfrac{1}{3!}(A^*)^3 + \dots \\ &= \exp(A^*)\end{aligned}$$

$\blacksquare$

#9.

Prove that if $A \in M_{n\times n}(\mathbb{C})$ satisfies $A^* = -A$ then $\exp(A)$ is unitary.

proof:

$$\exp(A)^{-1} = \exp(-A) = \exp(A^*) = \exp(A)^*.$$

Therefore, $\exp(A)$ is unitary.

$\blacksquare$

#10

Prove that if $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable then

$$\det(\exp(A)) = e^{\operatorname{Tr}(A)}.$$

proof:

Since A is diagonalizable there exists $V \in M_{n\times n}(\mathbb{C})$ and diagonal $\Lambda \in M_{n\times n}(\mathbb{C})$ such that

$$A = V\Lambda V^{-1}.$$

Therefore, if $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ it follows that

$$\exp(A) = V \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} V^{-1}$$

$$\Rightarrow \det(\exp(A)) = \det\left(V \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} V^{-1} \right)$$

$$= \det(V) \det\left(\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} \right) \det(V^{-1})$$

$$= \frac{\det(V)}{\det(V)} e^{\lambda_1} \cdots e^{\lambda_n}$$

$$= e^{\lambda_1 + \cdots + \lambda_n}$$

$$= e^{Tr(A)}.$$

$\blacksquare$

#11

Consider the inconsistent system $A\vec{x} = \vec{b}$ when

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

(a) Find the associated normal equations.

(b) Find the least square solution to the system.

Solution:

(a) $$A^* = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \Rightarrow A^*A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^*\vec{b} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

(b) Therefore

$$\hat{x} = \frac{1}{3} \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}.$$

$\blacksquare$

#13

A 20 lb turkey that is at room temperature of 72° is placed in the oven at 1:00 PM. The temperature of the turkey is observed in 20 minute intervals to be 79°, 88°, and 96°. A turkey is cooked when its temperature reaches 165°. Using least squares, estimate how much longer you have to wait until the turkey is done cooking.

Solution:

We have the following data points:

$$(0,72), (1,79), (2,88), (3,96)$$

where time is measured in units of 20 minutes. Assuming $T = mt + b$ we obtain

$$72 = b$$

$$79 = m + b$$

$$88 = m \cdot 2 + b$$

$$96 = m \cdot 3 + b$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b \\ m \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 72 \\ 79 \\ 88 \\ 96 \end{bmatrix}}_{\vec{b}}$$

$$\Rightarrow A^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \Rightarrow A^*A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \Rightarrow A^*\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 72 \\ 79 \\ 88 \\ 96 \end{bmatrix} = \begin{bmatrix} 335 \\ 543 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \hat{x} = \begin{bmatrix} 335 \\ 543 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 4 & 6 & 335 \\ 6 & 14 & 543 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 358/5 \\ 0 & 1 & 81/10 \end{array}\right]$$

The line of best fit

$$T = \frac{81}{10} t + \frac{358}{5}$$

Solving, we have that

$$165 = \frac{81}{10} t + \frac{358}{5} \Rightarrow \frac{1650 - 716}{81} = t \Rightarrow t \approx 11.5 = 230 \text{ minutes.}$$

The turkey should be done around 4:50 PM