MTH 225: Homework #4

Due Date: February 23, 2024

1. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(a_1, a_2, a_3) = (4a_1 + a_3, 2a_1 + 3a_2 + 2a_3, a_1 + 4a_3).$$

   (a) Let $\mathcal{S} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the standard basis of $\mathbb{R}^3$. Find $[T(\mathcal{S}, \mathcal{S})]$ the matrix of T with respect to $\mathcal{S}$.

   (b) Find the characteristic polynomial of T. Are all the roots in $\mathbb{R}$?

   (c) Find the eigenvalues of T.

   (d) Find a basis for the eigenspace of T corresponding to each eigenvalue. What are their dimensions?

   (e) Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ that consists of eigenvectors of T.

   (f) Find $[T(\mathcal{B}, \mathcal{B})]$, the matrix of T with respect to $\mathcal{B}$.

   (g) Find the matrix P so that $[T(\mathcal{B}, \mathcal{B})] = P^{-1}[T(\mathcal{S}, \mathcal{S})]P$

2. If $B = PAP^{-1}$, then prove $B^n = PA^nP^{-1}$ for any $n \in \mathbb{Z}$.

3. Suppose that A and B are $n \times n$ diagonalizable matrices with the same eigenspaces (but not necessarily the same eigenvalues). Prove that $AB = BA$.

4. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a set of *distinct* eigenvalues of T, and let $\{v_1, v_2, \ldots, v_k\}$ be a set of vectors such that v_i is an eigenvector corresponding to the eigenvalue λ_i. Prove that $\{v_1, v_2, \ldots, v_k\}$ is a set of linearly independent vectors.

5. Consider $e^x, e^{2x}, \ldots, e^{nx}$. Show that each of these functions in $\mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$ is an eigenvector for the differentiation operator. Here, $\mathbb{C}(\mathbb{R}, \mathbb{R})$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

6. Let $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\vec{v} = (v_1, \ldots v_n) \in \mathbb{R}^n$. Let $A = \vec{u}\vec{v}^T$

   (a) Find the columns of A in terms of $\vec{u}$ and $\vec{v}$.

   (b) Show that A is a rank 1 matrix.

7. Give an example of a matrix $A \in M_{4\times4}(\mathbb{R})$ such that $\text{im}(A) = \ker(A)$. Show that there does not exist a matrix $A \in M_{5\times5}(\mathbb{R})$ such that $\text{im}(A) = \ker(A)$.

8. Let $\vec{u}, \vec{v} \in \mathbb{C}^n$. **Hint: For this problem you might have to review the geometric interpretation of how vectors are added and subtracted**.

   (a) Prove that $\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

   (b) Prove that if $\vec{u}$ and $\vec{v}$ have the same norm, then $\vec{u} + \vec{v}$ is orthogonal to $\vec{u} - \vec{v}$.

   (c) Prove that the diagonals of a rhombus are orthogonal to each other.

   (d) Prove the following

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).$$

   (e) Prove that the sum of the squares of the length of the four sides of a parallelogram is equal to the sum of the squares of the length of the two diagonals.

9. If $\vec{v}_1 \ldots \vec{v}_n$ are mutually orthogonal nonzero vectors, prove that they must be linearly independent.