MTH 225: Homework #7

Due Date: March 22, 2024

1. Suppose that $A, B \in M_{n\times n}(\mathbb{C})$ are diagonalizable matrices with the same eigenspaces (but not necessarily the same eigenvalues). Prove that $AB = BA$.

2. Compute a unitary diagonalization of each of the following Hermitian matrices (give the diagonal matrix and the unitary matrix) and give the spectral decomposition:

$$A = \begin{bmatrix} 7 & i \\ -i & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1-i \\ 1+i & 0 \end{bmatrix}.$$

3. Consider the following vectors in $\mathbb{C}^4$:

$$\vec{w}_1 = \begin{bmatrix} 1 \\ i \\ -i \\ 1 \end{bmatrix}, \qquad \vec{w}_2 = \begin{bmatrix} i \\ 1 \\ 1 \\ -i \end{bmatrix}, \qquad \vec{w}_3 = \begin{bmatrix} -1 \\ i \\ -i \\ 1 \end{bmatrix}.$$

   (a) Show that $\langle \vec{w}_1, \vec{w}_2 \rangle = 0$.

   (b) Find an orthonormal basis of the subspace $W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ of $\mathbb{C}^4$.

4. Prove the converse of the Spectral Theorem: If $A = UDU^*$ for a unitary matrix U and a diagonal matrix D, whose entries are all real numbers, then A must be a Hermitian matrix.

5. Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{C}^n$.

   (a) Find a nonzero eigenvalue of $\vec{u}\vec{v}^*$, and determine its corresponding eigenvector.

   (b) Determine the unique nonzero singular value σ of $\vec{u}\vec{v}^*$, as well as the corresponding singular vectors $\vec{u}_1$ and $\vec{v}_1$ corresponding to σ.

6. Let $\{\vec{u}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Prove that for any $\vec{v} \in \mathbb{C}^n$, one has the equality

$$\|\vec{v}\|^2 = \sum_{j=1}^{n} |\langle \vec{u}_j, \vec{v} \rangle|^2.$$

   Hint: Use the projection formula to express $\vec{v}$ as a linear combination of the given basis.

7. A matrix $P \in M_{n\times n}(\mathbb{C})$ is called a projection matrix if $P^2 = P$.

   (a) Show that if $P \in M_{n\times n}(\mathbb{C})$ is a projection matrix and $P\vec{v} \neq \vec{v}$ then $P\vec{v} - \vec{v} \in \ker(A)$.

   (b) Prove that if $P \in M_{n\times n}(\mathbb{C})$ is a projection matrix then $I - P$ is also a projection matrix.

   (c) If $\vec{q} \in \mathbb{C}^n$ satisfies $\|\vec{q}\| = 1$, prove that $\vec{q}\vec{q}^*$ is a projection matrix.

   (d) If P is projection matrix, prove the following three statements

$$\begin{aligned} \text{im}(I-P) &= \ker(P), \\ \text{im}(P) &= \ker(I-P), \\ \text{im}(P) \cap \ker(P) &= \{0\}. \end{aligned}$$

8. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(a) Find the singular value decomposition of A, B, C. You will probably have to use the Gram Matrix to compute these decompositions.

(b) Compute the closest rank 1 matrices to A, B, and C.