MTH 225: Homework #9

Due Date: April 19, 2024

1. Consider the following quadratic form of real variables $\vec{x} = (x_1, x_2, x_3)$:

$$Q(\vec{x}) = Q(x_1, x_2, x_3) = 6x_1^2 - 4x_1x_2 - 2x_1x_3 + 6x_2^2 - 2x_2x_3 + 5x_3^2.$$

   (a) Express $Q(\vec{x})$ as $\vec{x}^T A\vec{x}$ for a symmetric matrix A.

   (b) Diagonalize Q (which is equivalent to diagonalizing A) by finding new variables y_1, y_2 and y_3 (express each in terms of x_1, x_2 and x_3), so that there are constants $a, b, c \in \mathbb{R}$ with $Q(x_1, x_2, x_3) = ay_1^2 + by_2^2 + cy_3^2$.

2. Consider the following quadratic form of real variables $\vec{x} = (x_1, x_2)$:

$$Q(\vec{x}) = Q(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 8x_2^2.$$

   (a) Express $Q(\vec{x})$ as $\vec{x}^T A\vec{x}$ for a symmetric matrix A.

   (b) Diagonalize Q and use it to graph the ellipse

$$5x^2 + 4xy + 8y^2 = 1.$$

3. For what values of $a, b, c \in \mathbb{R}$ is the the quadratic form

$$q(x, y, z) = x^2 + axy + y^2 + bxz + cyz + z^2$$

   nonnegative for all values of $(x, y, z) \in \mathbb{R}^3$.

4. (a) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is positive definite then $\det(A) > 0$.

   (b) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is positive definite then $\mathrm{Tr}(A) > 0$.

   (c) Prove that if $A \in M_{2\times 2}(\mathbb{C})$ is Hermitian and satisfies $\mathrm{Tr}(A) > 0$ and $\det(A) > 0$ then A is positive definite.

   (d) Find a symmetric matrix $A \in M_{3\times 3}(\mathbb{C})$ with positive determinant and positive trace that is not positive definite.

5. Show that $A \in M_{n\times n}$ and $A^T \in M_{n\times n}$ have the same characteristic polynomial.

6. Show that $A \in M_{n\times n}$ and $A^T \in M_{n\times n}$ have the same eigenvalues. Find a counterexample to show that they do not necessarily have the same eigenvectors.

7. The refined Gershgorin domain is given by $D_A^* = D_{A^T} \cap D_A$. Show that the eigenvalues of A must lie in the refined Gershgorin domain.

8. Show that if A is Hermitian, strictly diagonally dominant, and each diagonal entry is positive, then A is positive defininite.

9. Find an invertible matrix $A \in M_{2\times 2}(\mathbb{C})$ whose Gershgorin domain contains 0.

10. For each of the following matrices, (i) find the Gershgorin disks of A and A^T, (iI) plot the refined Gershgorin domain in the complex plane, (iii) compute the eigenvalues and confirm the truth of the Gershgorin circle theorem.

$$A = \begin{bmatrix} 1 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & 3 & -4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$