Lecture 1: Solving Linear Systems

Example:

Solve:

$$3x_1+5x_2-4x_3=7$$
$$-3x_1-2x_2+4x_3=-1$$
$$6x_1+x_2-8x_3=-4$$

Augmented Matrix:

$$\left[\begin{array}{ccc:c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right]\begin{array}{l} \\ +R1 \\ -2R1 \end{array} \Rightarrow \left[\begin{array}{ccc:c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right]\begin{array}{l} \\ /3 \\ +3R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} 3x_1+5x_2-4x_3=7 \\ x_2 = 2 \\ 0 = 0 \end{array}$$

$\Rightarrow x_3$ cab be any numbr

$x_2=2$

$\Rightarrow 3x_1+10-4t=7$

$\Rightarrow x_1=-1+\frac{4}{3}t$

The solution can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+\frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \frac{x_3}{3}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

where $t \in \mathbb{R}$.

*This is cumbersome to write. Vector spaces are a way to reduce complexity of solutions.

Example:

How are colors implemented in a computer?

→ A computer screen shoots combinations of red, blue, and green into your eye. In 16 bit color each pixel has 2^{16} variations of red, blue, and green. This means there are

$$2^{16} \cdot 2^{16} \cdot 2^{16} = 2^{48} \text{ (trillions of possible colors)}$$

→ Instead of defining a name for each color we take a combination of them

$$\text{Color} = c_1 \text{red} + c_2 \text{blue} + c_3 \text{green}$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{B_1} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{B_1} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{B_1} \qquad B_1 = \text{primary basis}$$

Color = vector space spanned by "red", "green", "blue".

→ When printing an image the basis of colors is magenta, cyan, yellow.

$$\text{Color} = d_1 \cdot \text{magenta} + d_2 \text{ cyan} + d_3 \text{ yellow}$$

$$= d_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{B_2} + d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{B_2} + d_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{B_2} \qquad B_2 = \text{printer basis}$$

How can we transform from what is observed on our screen to what is printed?
To do so, we will need to learn about change of basis and linear transformations.

Example:

For what values of a does the system of equations have zero, one, or infinitely many solutions?

$$x_1 + x_2 + x_3 = 4$$
$$x_2 = 2$$
$$(a^2-4)x_3 = a-2$$

If $a \neq \pm 2$ we have that

$$x_2 = 2$$
$$x_3 = \frac{a-2}{a^2-4} = \frac{1}{a+2} \Rightarrow \text{One solution}$$
$$x_1 = 2 - \frac{1}{a+2}. \Rightarrow \text{Solutions} = \left\{ v \in \mathbb{R}^3 : v = \begin{bmatrix} 2 - 1/(a+2) \\ 2 \\ 1/(a+2) \end{bmatrix} \right\}.$$

If $a = 2$ we have

$$x_2 = 2$$
$$0x_3 = 0 \Rightarrow x_3 = \text{anything}$$
$$x_1 = 2 - x_3$$
$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Solutions} = \left\{ v \in \mathbb{R}^3 : v = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$$

$\Rightarrow$ infinitely many solutions

If $a = -2$ we have

$$x_1 + x_2 + x_3 = 4$$
$$x_2 = 2$$
$$0x_3 = -4$$

$\Rightarrow \text{Solutions} = \emptyset$

$\Rightarrow$ no solutions.