Lecture 10: Unitary Matrices and the SVD

- If $A=(a_{ij})$ then $A^T=(a_{ji})$ is the transpose of A
- If $A=(a_{ij})$ then $A^*=(a_{ji}^*)$ is the conjugate transpose of A.
- A is called symmetric if $A^T=A$.
- A is called Hermitian if $A^*=A$

Definition- A matrix A is called unitary if the columns of A are orthonormal with respect to the complex inner product

Example:

$$A=\begin{bmatrix}(1+i)/2 & (1-i)/2\\ (1+i)/2 & (-1+i)/2\end{bmatrix}$$

$$\vec{q}_1=\begin{bmatrix}(1+i)/2\\ (1+i)/2\end{bmatrix},\quad \vec{q}_2=\begin{bmatrix}(1-i)/2\\ (-1+i)/2\end{bmatrix}$$

$$\Rightarrow \langle \vec{q}_1,\vec{q}_2\rangle=(1+i)/2\cdot(1+i)/2+(1+i)/2\,(-1-i)/2$$
$$=\frac{1+i}{4}(1+i-1-i)$$
$$=0$$

$$\langle \vec{q}_1,\vec{q}_1\rangle=(1+i)/2\,(1-i)/2+(1+i)/2\,(1-i)/2$$
$$=\tfrac{1}{4}(1+1+1+1)$$
$$=1$$

$$\langle \vec{q}_2,\vec{q}_2\rangle=(1-i)/2\,(1+i)/2+(-1+i)/2\,(-1-i)/2$$
$$=\tfrac{1}{4}(1+1+1+1)$$
$$=1$$

Therefore, $\{\vec{q}_1,\vec{q}_2\}$ form an orthonormal set $\Rightarrow A$ is unitary.

Theorem - A is unitary if and only if $A^* = A^{-1}$.

proof:

($\Rightarrow$) If A is unitary then there exists orthonormal $\{\vec{q}_1, \dots, \vec{q}_n\}$ such that

$$A = \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}$$

$$\Rightarrow A^*A = \begin{bmatrix} \vec{q}_1^* \\ \vdots \\ \vec{q}_n^* \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{q}_1^*\vec{q}_1 & \vec{q}_1^*\vec{q}_2 & \cdots & \vec{q}_1^*\vec{q}_n \\ \vdots & \vdots & & \vdots \\ \vec{q}_n^*\vec{q}_1 & \vec{q}_n^*\vec{q}_2 & \cdots & \vec{q}_n^*\vec{q}_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I$$

The same is true for $A \cdot A^*$, i.e., $A \cdot A^* = I$. Then $A^* = A^{-1}$.

($\Leftarrow$) If $A^* = A^{-1}$ then if we let $\vec{q}_i$ denote the columns of A then

$$A^*A = \begin{bmatrix} \vec{q}_1^* \\ \vdots \\ \vec{q}_n^* \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}$$

Theorem - For all $\vec{u}, \vec{v} \in \mathbb{C}^n$,

$$\langle \vec{u}, \vec{v} \rangle = \langle Q\vec{u}, Q\vec{v} \rangle$$

for the standard inner product.

proof:

$$\langle Q\vec{u}, Q\vec{v} \rangle = (Q\vec{u})^* Q\vec{v}$$
$$= \vec{u}^* Q^* Q \vec{v}$$
$$= \vec{u}^* \vec{v} = \langle \vec{u}, \vec{v} \rangle$$

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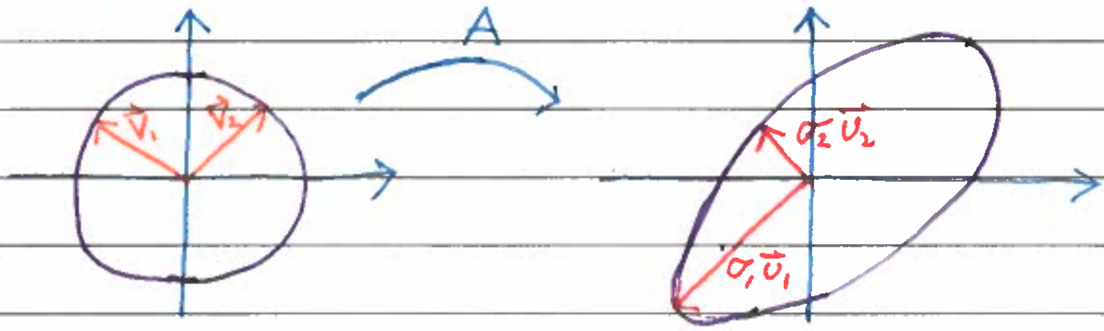
* From this theorem it follows that angles and norms are preserved. ($\|Q\vec{u}\| = \|\vec{u}\|$)

Singular Value Decomposition

- A matrix $A \in M_{m\times n}(\mathbb{R})$ rotates and stretches vectors.
  - $\Rightarrow$ To figure out a matrix we can operate on the unit sphere in $\mathbb{R}^n$



$$\Rightarrow \underbrace{\begin{bmatrix} A \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} \vec{u}_1 & \cdots & u_n \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}}_{\Sigma}$$

$$\Rightarrow \boxed{A = U\Sigma V^*} \rightarrow \text{SVD}$$

Theorem - Every matrix has an SVD and the singular values $\sigma_1, \ldots, \sigma_n$ are unique.

Finding the SVD

- $A^*A = V\Sigma U^* U \Sigma V^*$
  $= V\Sigma^2 V^*$

$\Rightarrow \vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors of A^*A

$\Rightarrow \sigma_1^2, \ldots, \sigma_n^2$ are eigenvalues of A^*A

$\Rightarrow \vec{u}_1 = A\vec{v}_1, \ldots, \vec{u}_n = A\vec{v}_n.$