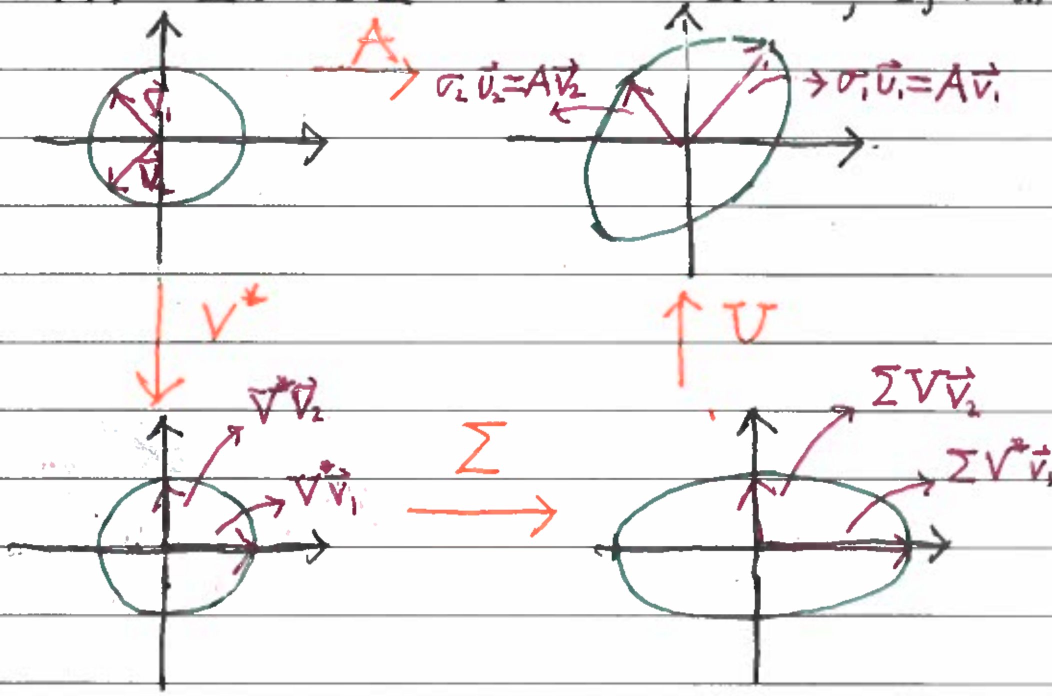
Lecture #11: Hermitian Matrices and the SVD

Geometric Interpretation of SVD:

$AV = \Sigma U = U\Sigma \Rightarrow A = U\Sigma V^*$, U, V are unitary



How do we show that matrices are nothing more than a composition of rotations/reflections, stretches, rotations/reflections??

Definition- A matrix $A \in M_{n\times n}(\mathbb{C})$ is Hermitian if for all $\vec{u}, \vec{v} \in \mathbb{C}^n$

$$\langle A\vec{u}, \vec{v}\rangle = \langle \vec{u}, A\vec{v}\rangle$$

Also called self adjoint.

Theorem- $A \in M_{n\times n}(\mathbb{C})$ is self adjoint if and only if

$$A^* = A.$$

~~Proof~~

A is self adjoint if and only if for all $\vec{u}, \vec{v} \in V$

$$\langle A\vec{u}, \vec{v}\rangle = \langle \vec{u}, A\vec{v}\rangle$$

$$\Leftrightarrow \vec{u}^* A^* \vec{v} = \vec{u}^* A\vec{v}$$

$$\Leftrightarrow A^* = A.$$

Theorem - If A is Hermitian then all of its eigenvalues are real.

proof:

Let $\vec{v}$ be an eigenvector of A with corresponding eigenvalue λ.

Therefore,

$$\langle A\vec{v}, \vec{v}\rangle = \langle \vec{v}, A\vec{v}\rangle$$

$$\Rightarrow \langle \lambda\vec{v}, \vec{v}\rangle = \langle \vec{v}, \lambda\vec{v}\rangle$$

$$\Rightarrow \bar{\lambda}\langle \vec{v}, \vec{v}\rangle = \lambda\langle \vec{v}, \vec{v}\rangle$$

$$\Rightarrow \bar{\lambda} = \lambda$$

$$\Rightarrow \lambda \text{ is real.}$$

Theorem - If A is hermitian, then eigenvectors of A corresponding to different eigenvalues are orthogonal.

proof

Let $\vec{u}, \vec{v} \in V$ be eigenvectors of A with distinct eigenvalues $\lambda, \mu \in \mathbb{R}$.

Consequently,

$$\langle A\vec{u}, \vec{v}\rangle = \langle \vec{u}, A\vec{v}\rangle$$

$$\Rightarrow \langle \lambda\vec{u}, \vec{v}\rangle = \langle \vec{u}, \mu\vec{v}\rangle$$

$$\Rightarrow \lambda\langle \vec{u}, \vec{v}\rangle = \mu\langle \vec{u}, \vec{v}\rangle$$

$$\Rightarrow (\lambda - \mu)\langle \vec{u}, \vec{v}\rangle = 0.$$

Since $\lambda \neq \mu$, $\langle \vec{u}, \vec{v}\rangle = 0$

Theorem - Let A be an $n \times n$ Hermitian matrix. Then there exists a unitary U and a diagonal matrix with real entries such that

$$A = U^* \Delta U.$$

Theorem - Let A be a rank k Hermitian matrix. There exists orthonormal vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ and real numbers $\lambda_1, \dots, \lambda_k$ such that

$$A = \lambda_1 \vec{v}_1 \vec{v}_1^* + \dots + \lambda_k \vec{v}_k \vec{v}_k^*$$

This decomposition is called the spectral decomposition of A.

proof:

Let $\vec{v}_1, \dots, \vec{v}_k$ be the eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_k$.

Let $B = \lambda_1 \vec{v}_1 \vec{v}_1^* + \dots + \lambda_k \vec{v}_k \vec{v}_k^*$. Consequently,

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad B\vec{v}_i = \lambda_1 \vec{v}_1 \vec{v}_1^* \vec{v}_i + \dots + \lambda_i \vec{v}_i \vec{v}_i^* \vec{v}_i + \dots + \lambda_k \vec{v}_k \vec{v}_k^* \vec{v}_i$$
$$= \lambda_i \vec{v}_i$$

$$\Longrightarrow A = B.$$

Return to SVD

Theorem - If $A \in M_{m \times n}(\mathbb{C})$ then the eigenvalues of A^*A are all nonnegative.

proof:

Let $\vec{v}$ be an eigenvector of A with corresponding eigenvalue λ.

Therefore,

$$\|A\vec{v}\|^2 = \langle A\vec{v}, A\vec{v} \rangle$$
$$= \langle \vec{v}, A^*A\vec{v} \rangle$$
$$= \langle \vec{v}, \lambda\vec{v} \rangle$$
$$= \lambda \langle \vec{v}, \vec{v} \rangle$$
$$= \lambda \|\vec{v}\|^2$$

$$\Longrightarrow \lambda \geq 0.$$

Theorem - Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A^*A with eigenvalues satisfying

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0$$

Then $\{A\vec{v}_1, \ldots, A\vec{v}_n\}$ is an orthogonal basis for $im(A)$ and $rank(A) = r$.

proof:

1. If $i \neq j$ we have that

$$\begin{aligned} \langle A\vec{v}_i, A\vec{v}_j \rangle &= \langle \vec{v}_i, A^*A\vec{v}_j \rangle \\ &= \langle \vec{v}_i, \lambda_j \vec{v}_j \rangle \\ &= \lambda_j \langle \vec{v}_i, \vec{v}_j \rangle \\ &= 0 \end{aligned}$$

since eigenvectors of A^*A are orthogonal.

2. Since $A\vec{v}_i = 0$ for $i > r$ and orthogonal vectors are linearly independent it follows that $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ are linearly independent.

3. Let $\vec{u} \in Im(A)$. Therefore, there exists $\vec{x} \in \mathbb{C}^n$ such that

$$\vec{u} = A\vec{x}.$$

Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis it follows that there exists $c_1, \ldots, c_n \in \mathbb{C}$ such that

$$\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

$$\begin{aligned} \Rightarrow \vec{u} &= A\vec{x} \\ &= A(c_1\vec{v}_1 + \ldots + c_n\vec{v}_n) \\ &= c_1 A\vec{v}_1 + \ldots + c_r A\vec{v}_r + c_{r+1} A\vec{v}_{r+1} + \ldots + c_n A\vec{v}_n \\ &= c_1 A\vec{v}_1 + \ldots + c_r A\vec{v}_r \end{aligned}$$

Therefore,

$$\vec{u} \in span\{A\vec{v}_1, \ldots, A\vec{v}_r\}.$$

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Theorem - Let $A \in M_{m\times n}(\mathbb{C})$ then there exists unitary matrices $U \in M_{m\times m}(\mathbb{C})$, $V \in M_{n\times n}(\mathbb{C})$ such that -

$$A = U\Sigma V^*$$

where,

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$ and r is the rank of A.

~~proof~~ -

Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis of eigenvectors of A^*A. with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Consequently, $\{A\vec{v}_1, \dots, A\vec{v}_n\}$ is an orthogonal basis for $im(A)$. Consequently, letting

$$\vec{U}_i = \frac{1}{\|A\vec{v}_i\|} A\vec{v}_i = \frac{1}{\sigma_i} A\vec{v}_i$$

it follows that $\{\vec{U}_1, \dots, \vec{U}_r\}$ is an orthonormal basis for $im(A)$. Extending $\{\vec{U}_1, \dots, \vec{U}_m\}$ to an orthonormal basis for $\mathbb{C}^m$ and define

$$U = [\vec{U}_1 | \cdots | \vec{U}_m], \quad V = [\vec{V}_1, \dots, \vec{V}_n]$$

it follows that

$$AV = U\Sigma$$
$$\Rightarrow A = U\Sigma V^*$$

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Corollary -

1. Eigenvalues of A^*A and AA^* are squares of the singular values.
2. Eigenvectors of AA^* are the $\vec{U}$ vectors
3. Eigenvectors of A^*A are the $\vec{V}$ vectors.

~~proof~~:

$$A^*A = (U\Sigma V^*)^*(U\Sigma V^*) = V\Sigma^* U^* U\Sigma V^* = V\Sigma^2 V^*$$