Lecture 12: SVD Calculation

Lets find the SVD for the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

If $A = U\Sigma V^*$ then $A^*A = V\Sigma U^* U^* \Sigma V^* = V\Sigma^2 V^*$.
Consequently, the eigenvectors of A^*A are the $\vec{v}_i$ vectors with corresponding eigenvalues σ_i^2.

$$\Rightarrow A^*A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A^*A) = \det\begin{pmatrix} \lambda-1 & -1 \\ -1 & \lambda-2 \end{pmatrix} = (\lambda-1)(\lambda-2)-1$$

$$\Rightarrow \det(\lambda I - A^*A) = \lambda^2 - 3\lambda + 1$$

Therefore, the eigenvalues of A^*A are given by

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2}$$
$$= \frac{3 \pm \sqrt{5}}{2}.$$

$\lambda_1 = (3+\sqrt{5})/2$:

$$\lambda I - A^*A = \begin{bmatrix} (1+\sqrt{5})/2 & -1 \\ -1 & (-1+5)/2 \end{bmatrix}$$

$$\begin{bmatrix} (1+\sqrt{5})/2 & -1 \\ -1 & (-1+\sqrt{5})/2 \end{bmatrix} \times (1-\sqrt{5})/2 = \begin{bmatrix} -1 & (-1+\sqrt{5})/2 \\ -1 & (-1+\sqrt{5})/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & (1-\sqrt{5})/2 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \tilde{v}_1 = \begin{bmatrix} (-1+\sqrt{5})/2 \\ 1 \end{bmatrix}$$

$$\|\vec{v}_1\|^2 = \frac{(-1+\sqrt{5})^2}{4} + 1 = \frac{6-2\sqrt{5}+4}{4} = \frac{10-2\sqrt{5}}{4} = \frac{5-\sqrt{5}}{2}$$

$$\Rightarrow \|\vec{v}_1\| = \sqrt{\frac{5-\sqrt{5}}{2}}$$

$$\Rightarrow \vec{v}_1 = \sqrt{\frac{2}{5-\sqrt{5}}} \begin{bmatrix} (-1+\sqrt{5})/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} (-1+\sqrt{5})/\sqrt{10-2\sqrt{5}} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix}$$

$\lambda_2 = (3-\sqrt{5})/2$:

Since $\vec{v}_2$ is orthogonal to $\vec{v}_1$ we obtain

$$\vec{v}_2 = \begin{bmatrix} (-1+\sqrt{5})/\sqrt{10-2\sqrt{5}} \\ -\sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix}$$

The singular values are given by

$$\sigma_1 = \sqrt{3+\sqrt{5}}/\sqrt{2}$$

$$\sigma_2 = \sqrt{3-\sqrt{5}}/\sqrt{2}$$

We know that

$$\sigma_1 \vec{u}_1 = A\vec{v}_1 \Rightarrow \vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1, \quad \sigma_2 \vec{u}_2 = A\vec{v}_2 \Rightarrow \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2$$

Therefore,

$$A\vec{v}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (-1+\sqrt{5})/\sqrt{2}\sqrt{5-\sqrt{5}} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix} = \begin{bmatrix} (1+\sqrt{5})/(\sqrt{10-2\sqrt{5}}) \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix}$$

$$\Rightarrow \vec{u}_1 = \frac{\sqrt{2}}{\sqrt{3+\sqrt{5}}} \begin{bmatrix} (1+\sqrt{5})/\sqrt{10-2\sqrt{5}} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix}$$

$$A\vec{v}_2 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}(-1+\sqrt{5})/\sqrt{10-2\sqrt{5}}\\ -\sqrt{2}/\sqrt{5-\sqrt{5}}\end{bmatrix}$$

$$= \begin{bmatrix}(-3+\sqrt{5})/\sqrt{10-2\sqrt{5}}\\ -\sqrt{2}/\sqrt{5-\sqrt{5}}\end{bmatrix}$$

$$\Rightarrow \vec{u}_2 = \frac{\sqrt{2}}{\sqrt{3-\sqrt{5}}}\begin{bmatrix}-3+\sqrt{5}/\sqrt{10-2\sqrt{5}}\\ -\sqrt{2}/\sqrt{5-\sqrt{5}}\end{bmatrix}$$

$$\Rightarrow A = U\Sigma V^* = [\vec{u}_1 | \vec{u}_2]\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}\begin{bmatrix}\vec{v}_1^{\,*}\\ \hline \vec{v}_2^{\,*}\end{bmatrix} = [\vec{u}_1 | \vec{u}_2]\begin{bmatrix}\sigma_1\vec{v}_1^{\,*}\\ \hline \sigma_2\vec{v}_2^{\,*}\end{bmatrix}$$

$= \sigma_1 \vec{u}_1\vec{v}_1^{\,*} + \sigma_2\vec{u}_2\vec{v}_2^{\,*}$ ← sum of rank 1 matrices.

Example:

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$\mathrm{im}(A) = \mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$, $\ker(A) = \mathrm{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

$\vec{u}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\sigma_1 = 1$, $\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, $\sigma_2 = 0$, $\vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Example:

$$A = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$$

$\mathrm{im}(A) = \mathrm{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ $\ker(A) = \mathrm{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$

$\Rightarrow \vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\sigma_1 = \sqrt{2}$, $\vec{u}_1 = \begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \sigma_2 = 0, \ \vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathrm{im}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \ \ker(A) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \ \sigma_2 = 0$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \ A\vec{v}_1 = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = 2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \ \sigma_1 = 2, \ \vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$