Lecture 13: Square Roots and Exponentials

Definition - $A \in M_{n\times n}(\mathbb{C})$ is called positive semidefinite if A is Hermitian and for all $\vec{v} \in \mathbb{C}^n$

$$\vec{v}^* A \vec{v} \geq 0.$$

A is called positive definite if $\vec{v}^* A \vec{v} = 0 \Leftrightarrow \vec{v} = 0$.

Theorem: If $A \in M_{n\times n}(\mathbb{C})$ is positive definite then the mapping $\langle \cdot, \cdot \rangle_A$ defined by

$$\langle \vec{x}, \vec{y} \rangle_A = \langle \vec{x}, A\vec{y} \rangle$$

is an inner product.

proof:

1. $$\begin{aligned} \langle \vec{x}, \vec{y} \rangle_A &= \overline{\langle \vec{x}, A\vec{y} \rangle} \\ &= \overline{\langle A\vec{y}, \vec{x} \rangle} \\ &= \langle \vec{y}, A\vec{x} \rangle \\ &= \langle \vec{y}, \vec{x} \rangle_A \end{aligned}$$

2. $$\begin{aligned} \langle \vec{x}+\vec{y}, \vec{z} \rangle_A &= \langle \vec{x}+\vec{y}, A\vec{z} \rangle \\ &= \langle \vec{x}, A\vec{z} \rangle + \langle \vec{y}, A\vec{z} \rangle \\ &= \langle \vec{x}, \vec{z} \rangle_A + \langle \vec{y}, \vec{z} \rangle_A \end{aligned}$$

3. $$\begin{aligned} \langle \lambda\vec{x}, \vec{y} \rangle_A &= \langle \lambda\vec{x}, A\vec{y} \rangle \\ &= \lambda \langle \vec{x}, A\vec{y} \rangle \\ &= \lambda \langle \vec{x}, \vec{y} \rangle_A. \end{aligned}$$

4. $$\begin{aligned} \langle \vec{x}, \vec{x} \rangle_A &= \langle \vec{x}, A\vec{x} \rangle \\ &\geq 0. \end{aligned}$$

Theorem - Let A be positive semidefinite with diagonalization

$$A = UDU^*.$$

Then $\sqrt{A} = UD^{1/2}U^*$.

proof:

$$\sqrt{A}\cdot\sqrt{A} = UD^{1/2}U^*UD^{1/2}U^* = UD^{1/2}D^{1/2}U^* = UDU^*.$$

Polar Form - Let $A \in M_{n\times n}(\mathbb{C})$. Then there exists a unitary $Q \in M_{n\times n}(\mathbb{C})$ such that

$$A = Q\sqrt{A^*A}$$

proof:

Let A have the SVD, $A = U\Sigma V^*$. Let $Q = UV^*$ and $M = V\Sigma V^*$

$$\Rightarrow QM = UV^*V\Sigma V^* = U\Sigma V^* = A$$

Furthermore,

$$M^2 = V\Sigma V^* V\Sigma V^* = V\Sigma^2 V^* = A^*A$$

$$\Rightarrow \sqrt{A^*A} = M.$$

$\blacksquare$

Definition - If $A \in M_{n\times n}(\mathbb{C})$ then

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \dots$$

Example:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Find $\exp(A)$. We can diagonalize to simplify this process

$$Tr(A) = 2+2 = \lambda_1 + \lambda_2$$

$$\Rightarrow \lambda_1 + \lambda_2 = 4$$

$$Det(A) = 5 = \lambda_1\lambda_2$$

$$\Rightarrow \lambda_1 = 4 - \lambda_2$$

$$\Rightarrow 5 = (4-\lambda_2)\lambda_2$$

$$\Rightarrow 5 = 4\lambda_2 - \lambda_2^2$$

$$\Rightarrow \lambda_2^2 - 4\lambda_2 + 5 = 0$$

$$\Rightarrow \lambda_2 = \frac{4 \pm \sqrt{16-20}}{2}$$

$$= 2 \pm i$$

$\lambda_1 = 2+i$:

$$\lambda_1 I - A = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{matrix} \\ -iR1 \end{matrix} \Rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\lambda_2 = 2-i$:

$$\lambda_2 I - A = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{matrix} \\ +iR1 \end{matrix} \Rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Now, $A = V \Lambda V^{-1}$

$$\Rightarrow \exp(A) = I + V\Lambda V^{-1} + \frac{1}{2} V \Lambda^2 V^{-1} + \frac{1}{3!} V \Lambda^3 V^{-1} + \dots$$

$$= V\left(I + \Lambda + \frac{1}{2}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \dots\right)V^{-1}$$

$$= V\begin{pmatrix} 1+\lambda_1+\frac{1}{2}\lambda_1^2+\frac{1}{3!}\lambda_1^3+\dots & 0 \\ 0 & 1+\lambda_2+\frac{1}{2}\lambda_2^2+\frac{1}{3!}\lambda_2^3+\dots \end{pmatrix}V^{-1}$$

$$= V\begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}V^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{2+i} & 0 \\ 0 & e^{2-i} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1}$$

$$= e^2\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1}$$

$$= e^2\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} \cos(1)+i\sin(1) & 0 \\ 0 & \cos(1)-i\sin(1) \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \frac{e^2}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} \cos(1)+i\sin(1) & i\cos(1)-\sin(1) \\ \cos(1)-i\sin(1) & -i\cos(1)-\sin(1) \end{bmatrix}$$

$$= \frac{e^2}{2}\begin{bmatrix} 2\cos(1) & -2\sin(1) \\ 2\sin(1) & 2\cos(1) \end{bmatrix}$$

$$= \begin{bmatrix} e^2\cos(1) & -e^2\sin(1) \\ e^2\sin(1) & e^2\cos(1) \end{bmatrix}$$

Theorem - If U is unitary, then $U = e^{iH}$ for some Hermitian matrix H.

~~proof~~:

Since U is $\overset{\text{unitarily}}{\downarrow}$ diagonalizable with eigenvalues λ_j satisfying $|\lambda_j| = 1$ there exists θ_j such that $\lambda_j = e^{i\theta_j}$. Therefore,

$$U = R \Lambda R^*$$

$$= R \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} R^*$$

$$= R \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} R^*$$

$$=: \exp\left(R \begin{bmatrix} i\theta_1 & & 0 \\ & \ddots & \\ 0 & & i\theta_n \end{bmatrix} R^* \right)$$

$$= \exp\left(i R \begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix} R^* \right)$$

$$\Rightarrow H = R \begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix} R^*$$

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Theorem - Let A be an $n \times n$ matrix with complex entries. There exists a Hermitian matrix H such that

$$A = e^{iH} \sqrt{A^* A}.$$